Sistemi Intelligenti Avanzati
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Università degli Studi di Milano

Search algorithms for planning

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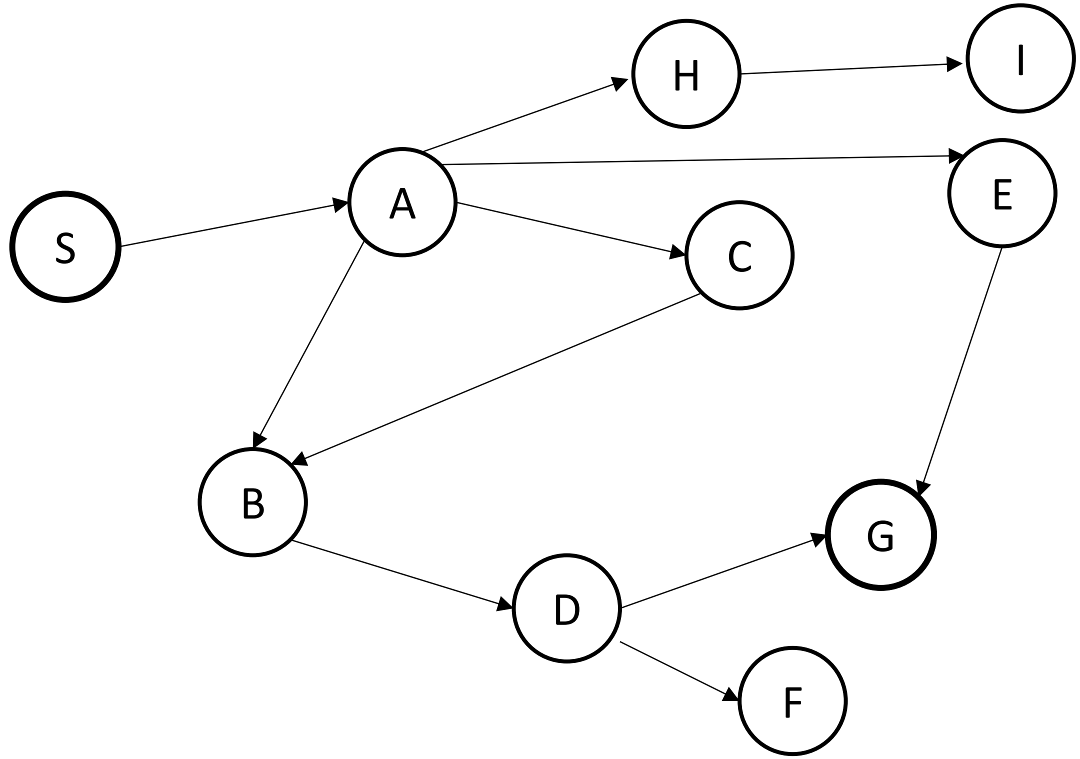
Search

Setting:

- Agent
- Goal
- Problem Formulation
  - A Set of Actions
  - A Set of States



What we want to do?

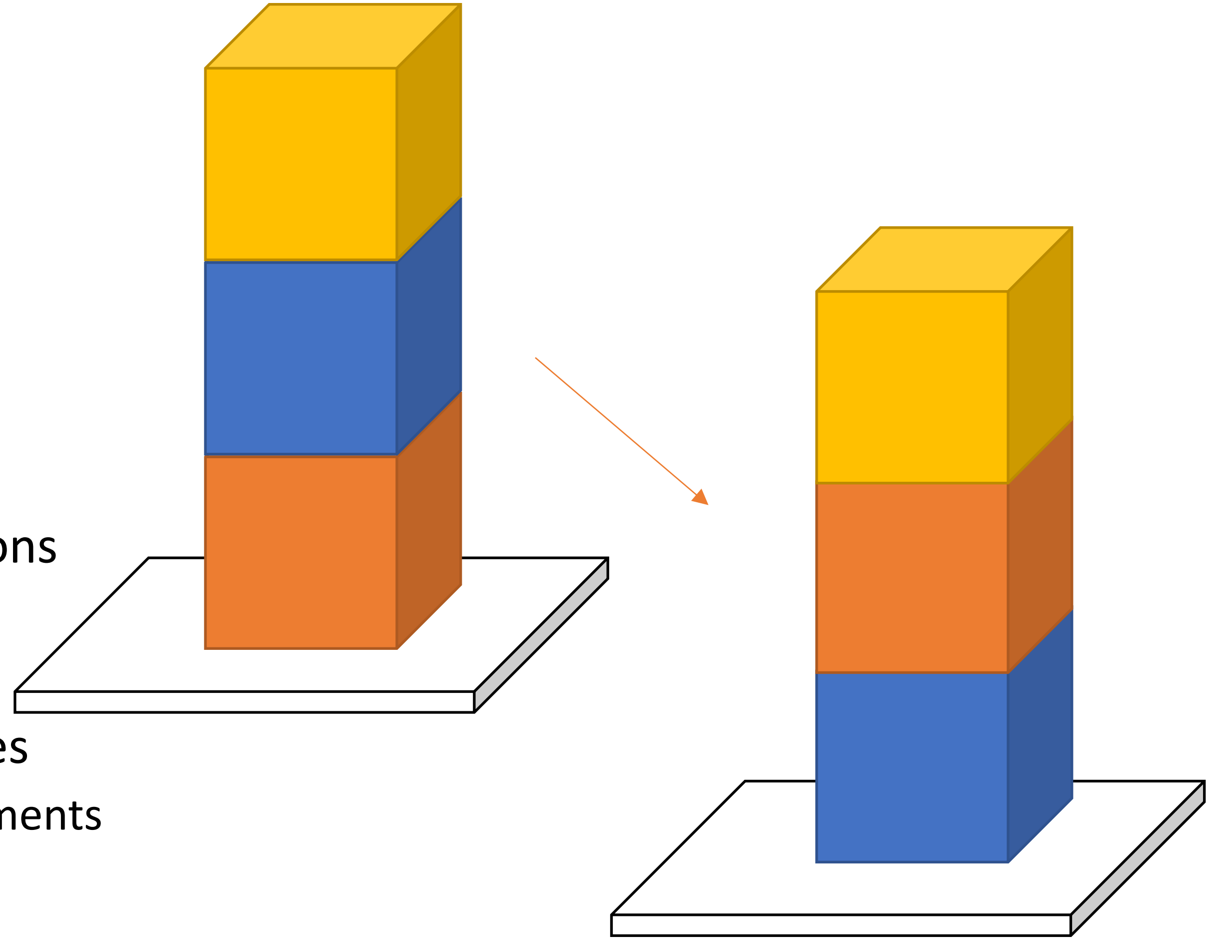
Find a set of actions that achieve the goal

when no single action will do

Planning

Setting:

- Agent
- Goal
- Problem Formulation
  - A Complex Set of Actions
    - Preconditions
    - Effects
  - A Complex Set of States
    - Propositional Statements



What we want to do?

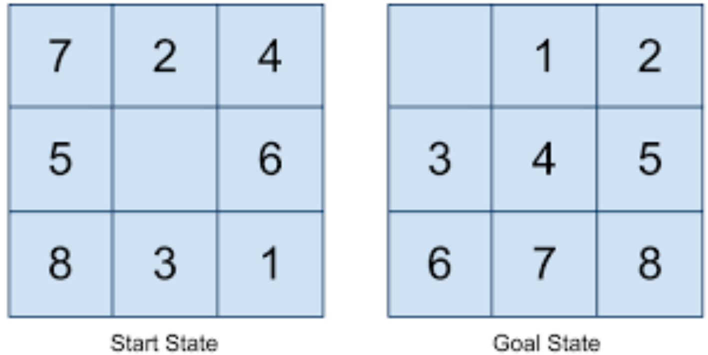
Take advantage of the structure of a problem

to construct complex plans of actions

Search algorithms for Planning

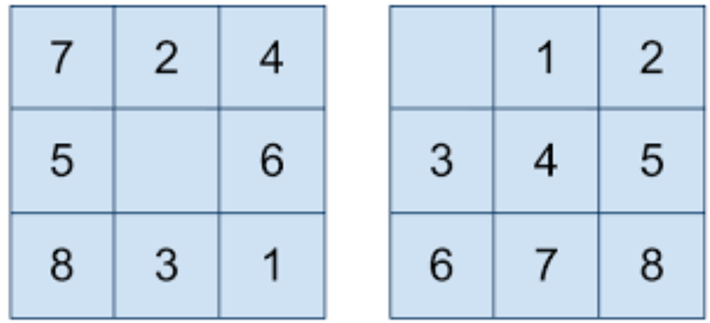
- Search and Planning often addresses similar problems and there is no clear distinction between them.
- On one hand, planning deals with problems where actions, states, goals cannot be described in a compact way, to have an abstract and high-level problem formulation.
- As an example, if the conditions can change planning methods are more suited to *adapt* the plan.
- On the other hand, search algorithms are often used when it is easier to describe the problem in a “mathematical” and compact way.
- Overall, search and planning are deeply connected and overlapped, and planning often requires some form of search and problem-solving algorithms.
- Path-planning is one of those problem.

Discrete Search Problems: 8-Puzzle



- States: location of each digits in the eight tiles + blank one
- Initial State
- Goal State
- Actions: Left, Right, Up, Down
- Transition: given a state and an action, the resulting board

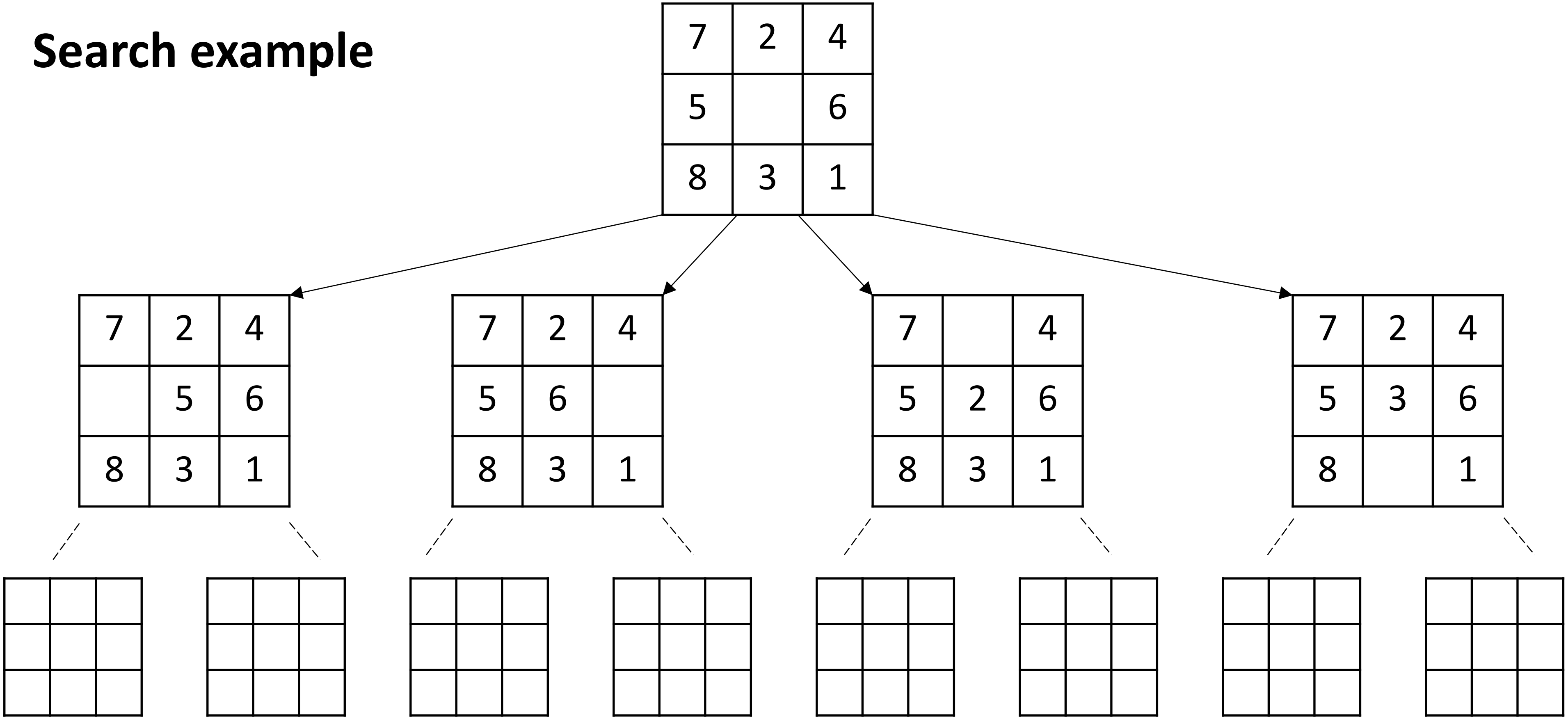
Discrete Search Problems: 8-Puzzle



Start State Goal State

- States: location of each digits in the eight tiles + blank one
- Initial State
- Goal State
- Actions: Left, Right, Up, Down
- Transition: given a state and an action, the resulting board
- Goal Test: if the states are equal to the goal state
- Cost: each movement costs 1, the lowest number of tile move the lowest the cost

Search example

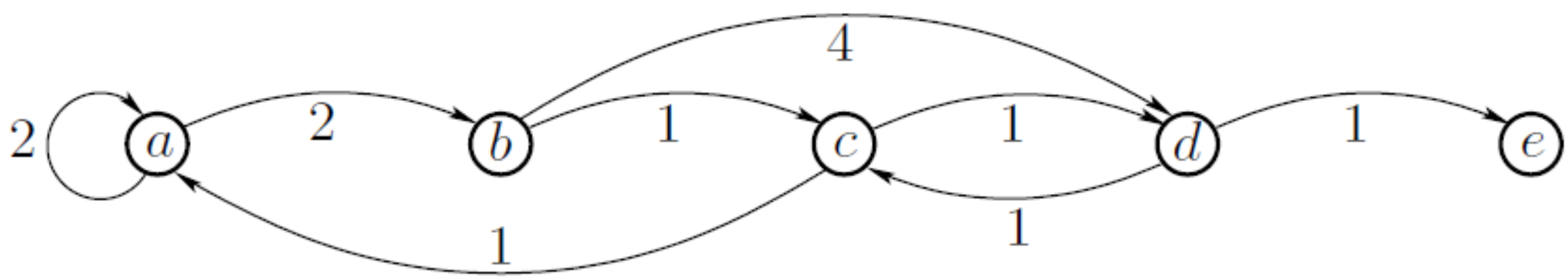


Expanding the current state by applying a legal action generating a new set of states, then…

…following up one option and putting aside others in case the first choice does not lead to a solution

State-based problem formulation

- State space defined as a set of **nodes**, each node represents a state; we assume a finite state space (and discrete)
- For each state, we have set of actions that can be undertaken by the agent from that state
- Transition model: given a starting state and an action, indicates an arrival state; we assume no uncertainties, i.e., deterministic transitions and full observability
- Action costs: any transition has a cost, which we assume to be greater than a positive constant (reasonable assumption, useful for deriving some properties of the algorithms we discuss)
- Initial state
- Goal State



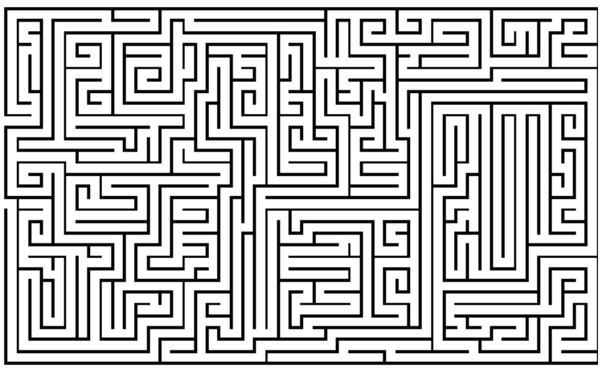
Compact representation: state transition graph G=(V,E)
(We will use "state" and "node" as interchangeable terms)

Formally describing the desired solution

- In the problem formulation we need to formally describe the features of the solution we seek

- Two (three) classes of problems:

feasibility

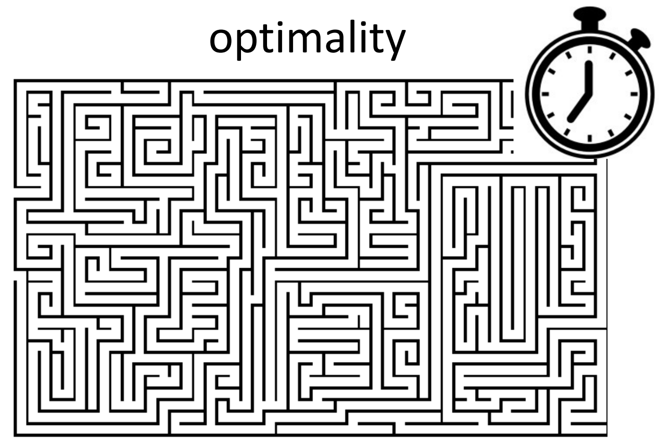


is there a path to an exit?

Set of goal states, find any sequence of actions (path) from the initial state to a goal state

(approximation)

optimality



If at least a path to an exit exists, what is the one with the minimum number of turns?

Set of goal states, find the sequence of actions (path) from the initial state to a goal state that has the minimum cost

Problem example

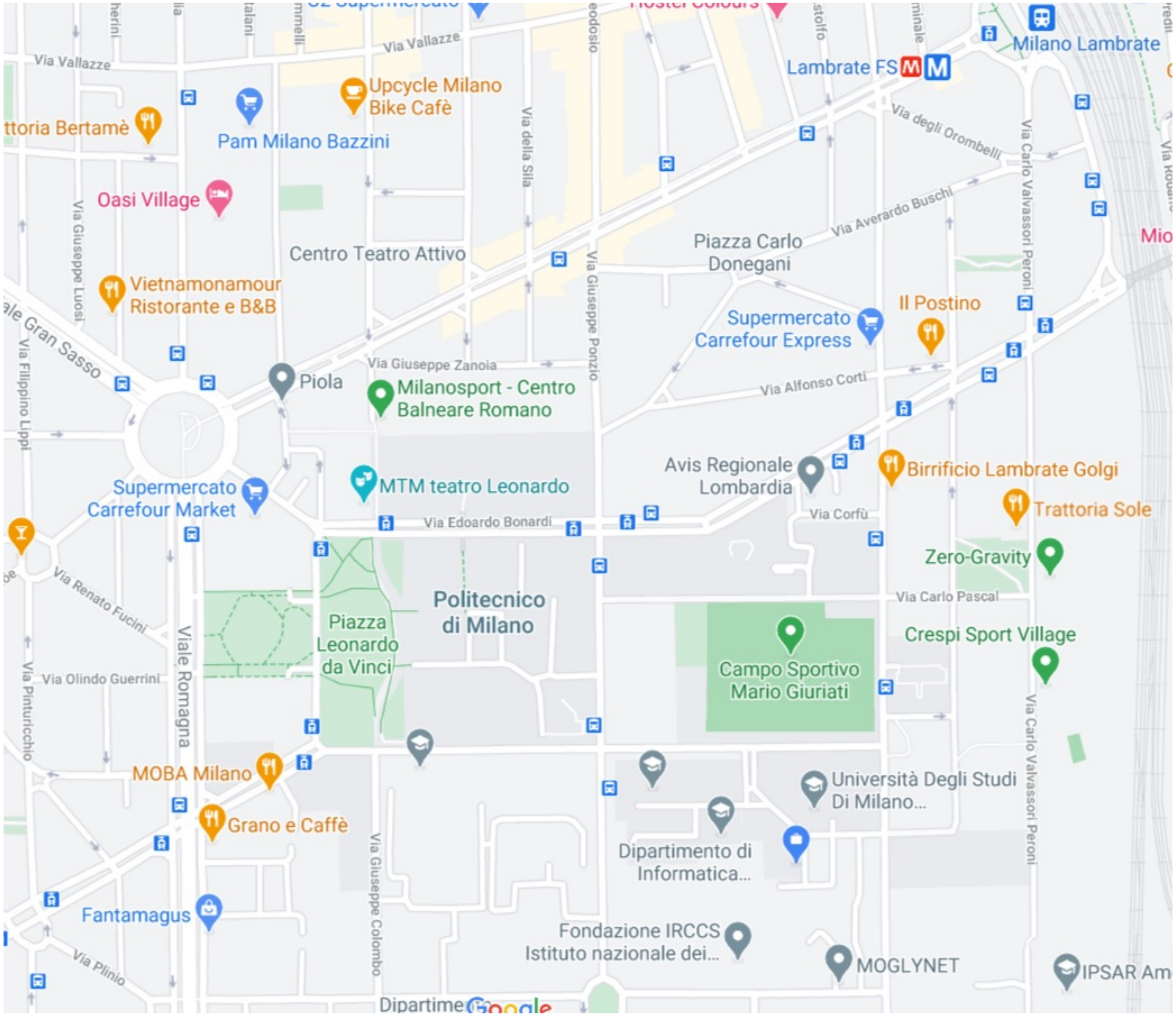
Consider a agent moving on a graph-represented environment:

- **States**: nodes of the graph, they represent physical locations
- **Edges**: represent connections between nearby locations or, equivalently, movement actions
- **Initial state**: some starting location for the agent

Desired solution:

- **Goal state(s)**: some location(s) to reach, …
  Find a path to the initial location to a goal one

Example: going home from the Celoria 18 with METRO



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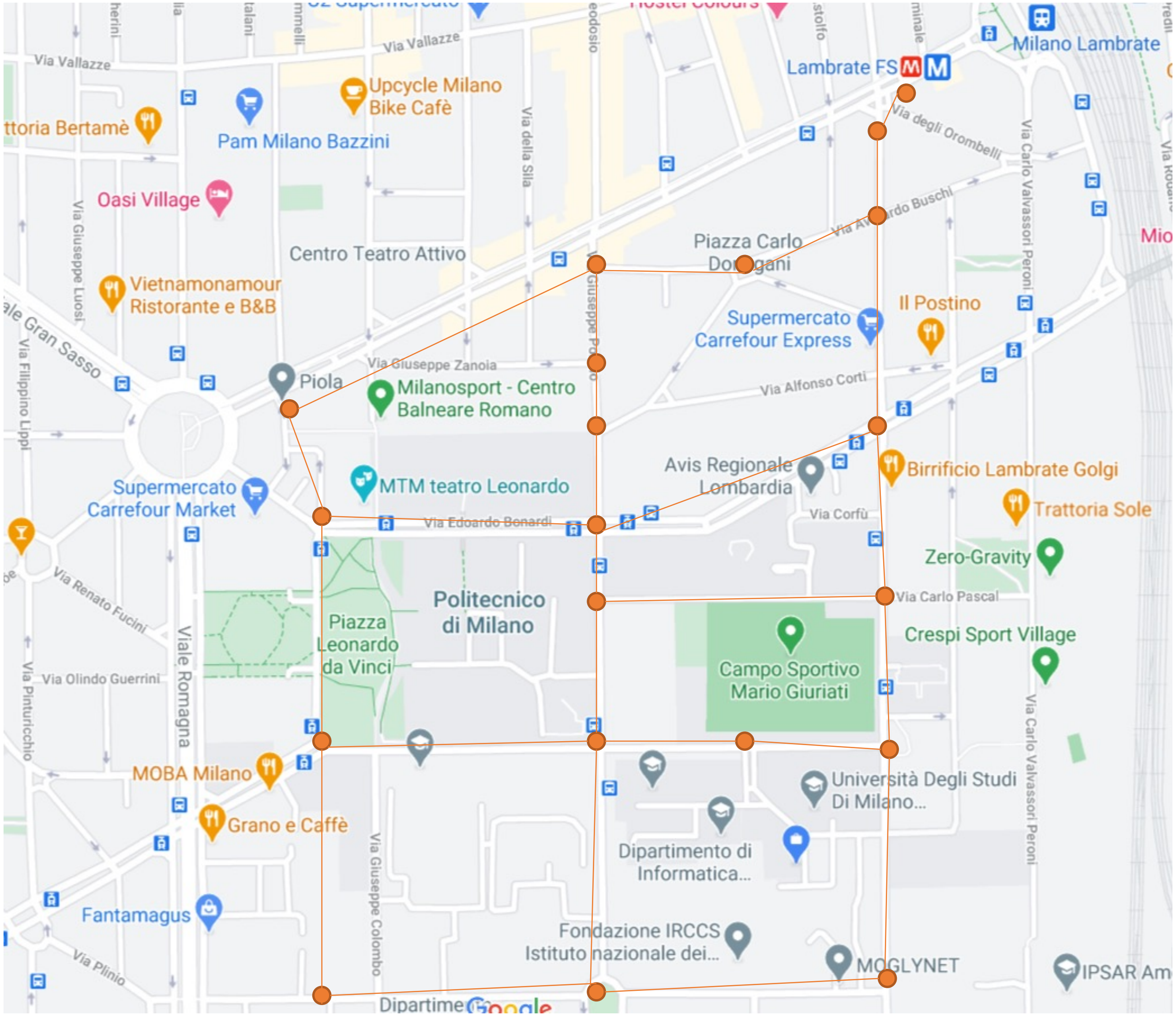

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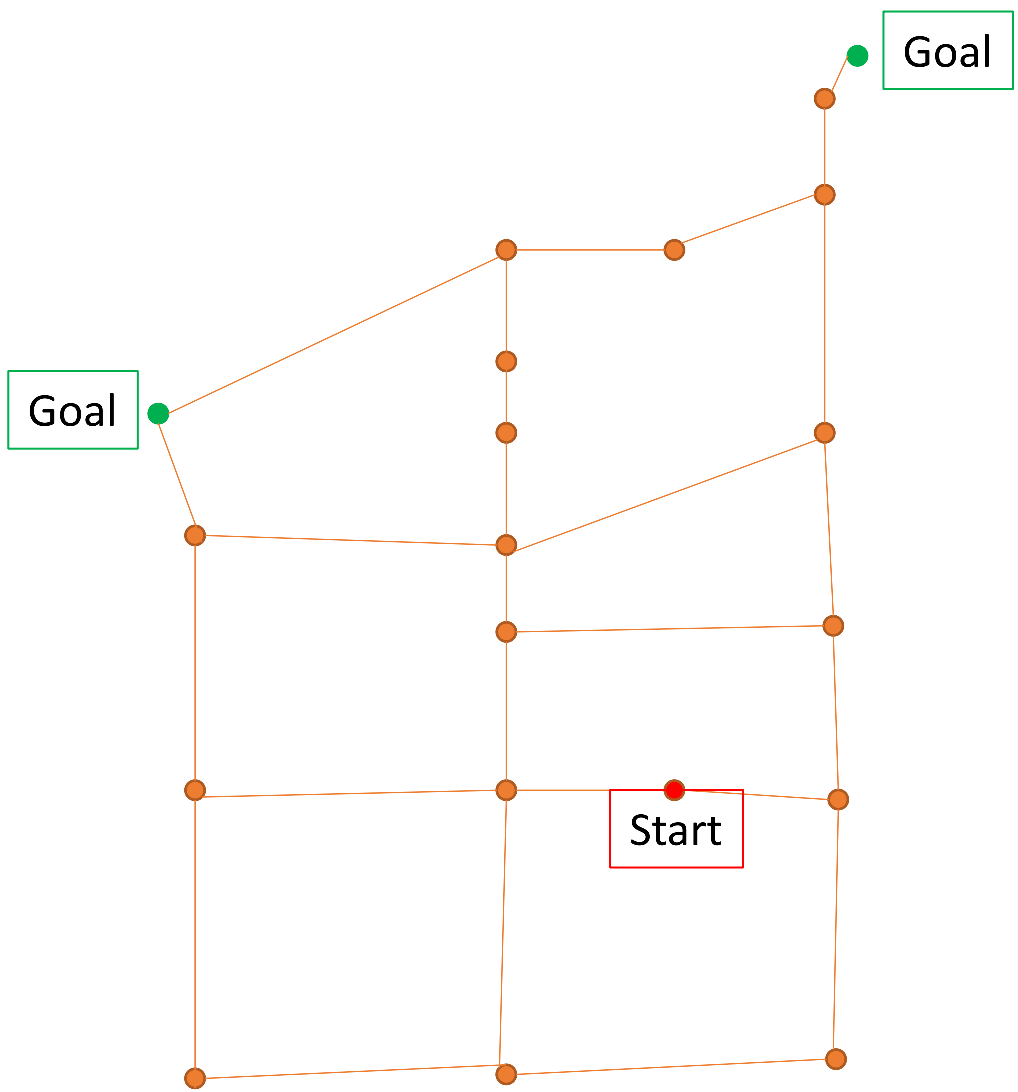


Problem example

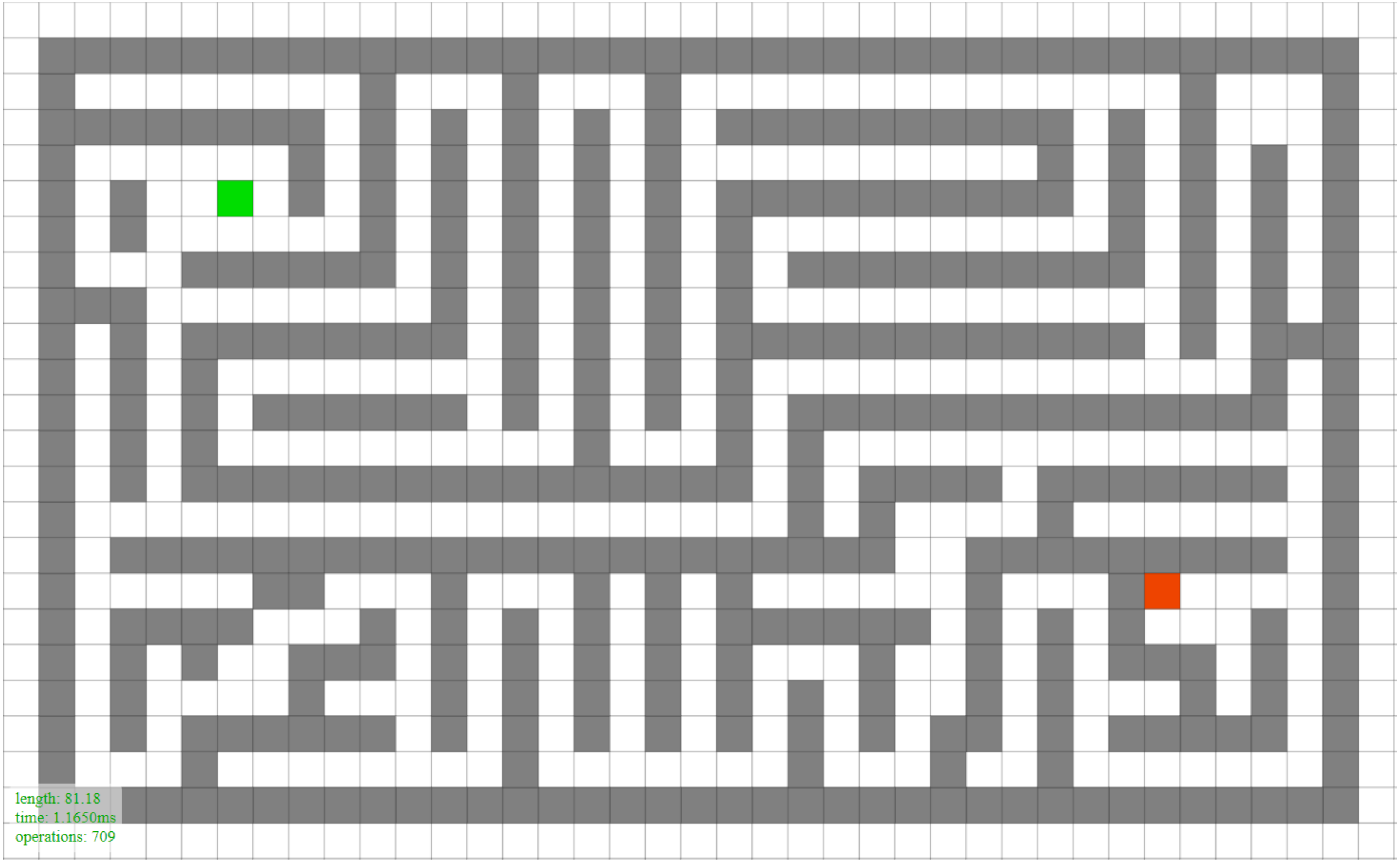
Consider a mobile robot moving on a grid environment:

- **States**: cells in the map, they represent physical locations
- **Edges**: represent connections between nearby locations or, equivalently, movement actions
- **Initial state**: some starting location for the robot

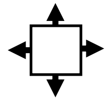
Desired solution:

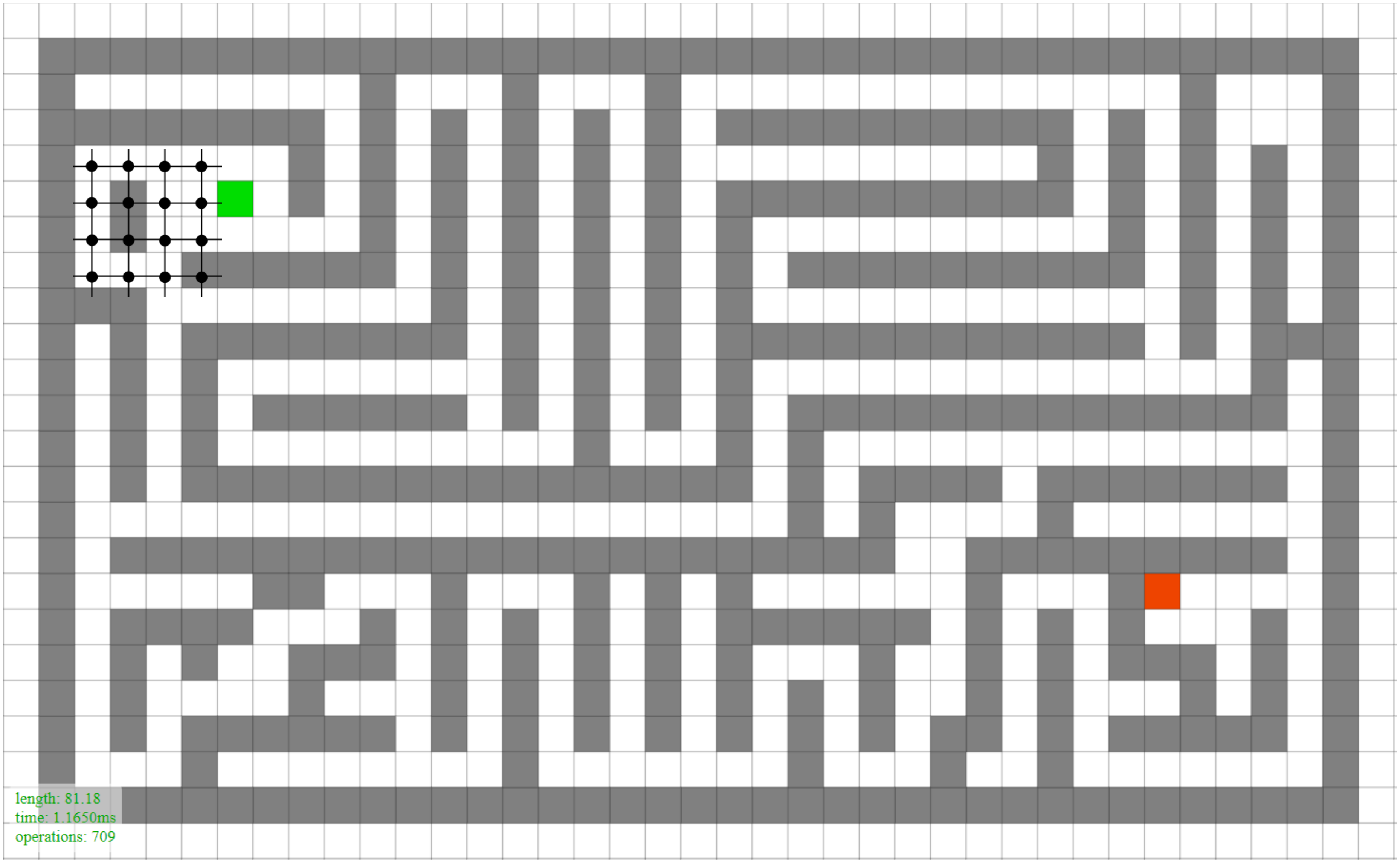
- **Goal state(s)**: some location(s) to reach
- Find a path to the initial location to a goal one

Problem Example

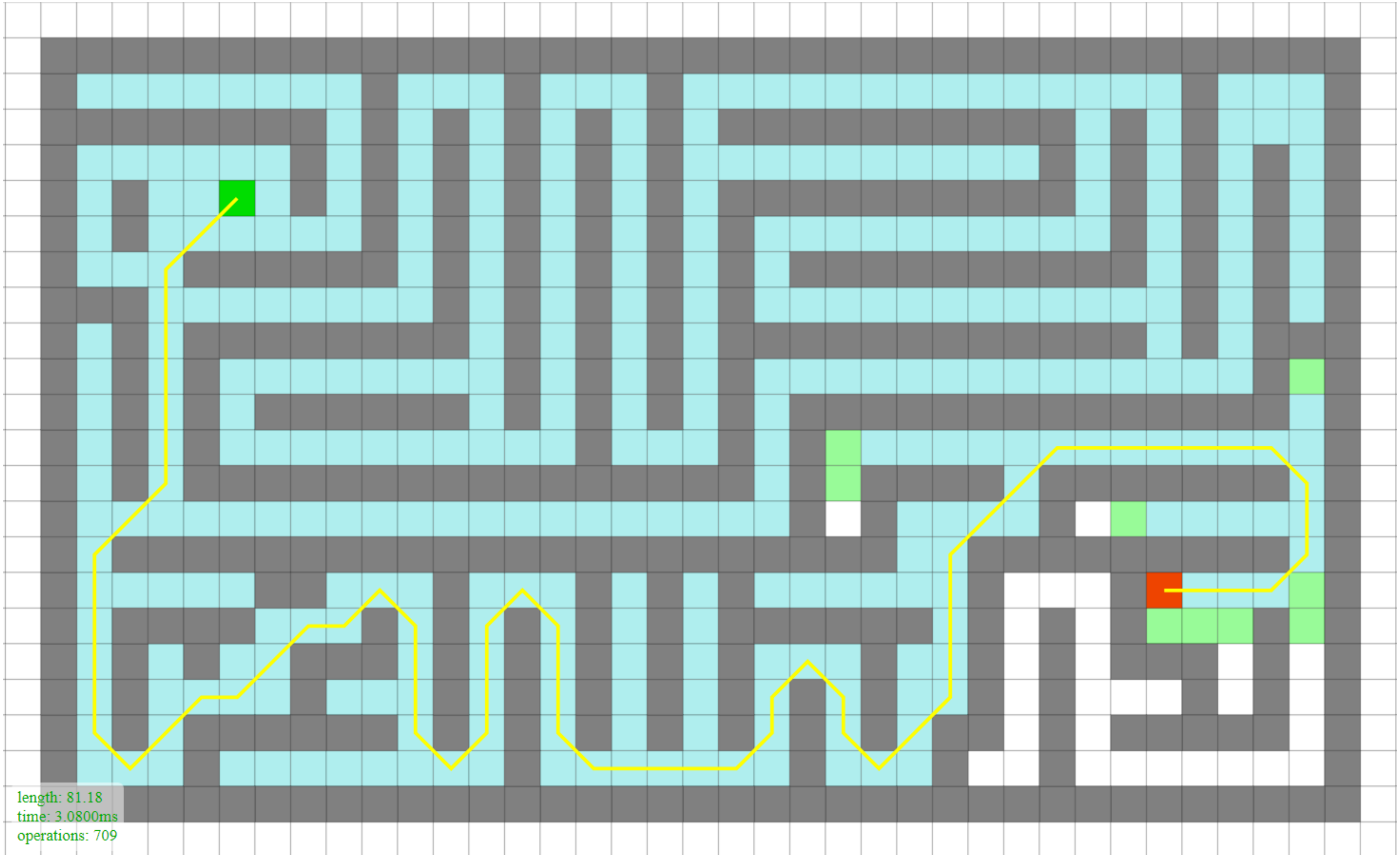


Problem Example

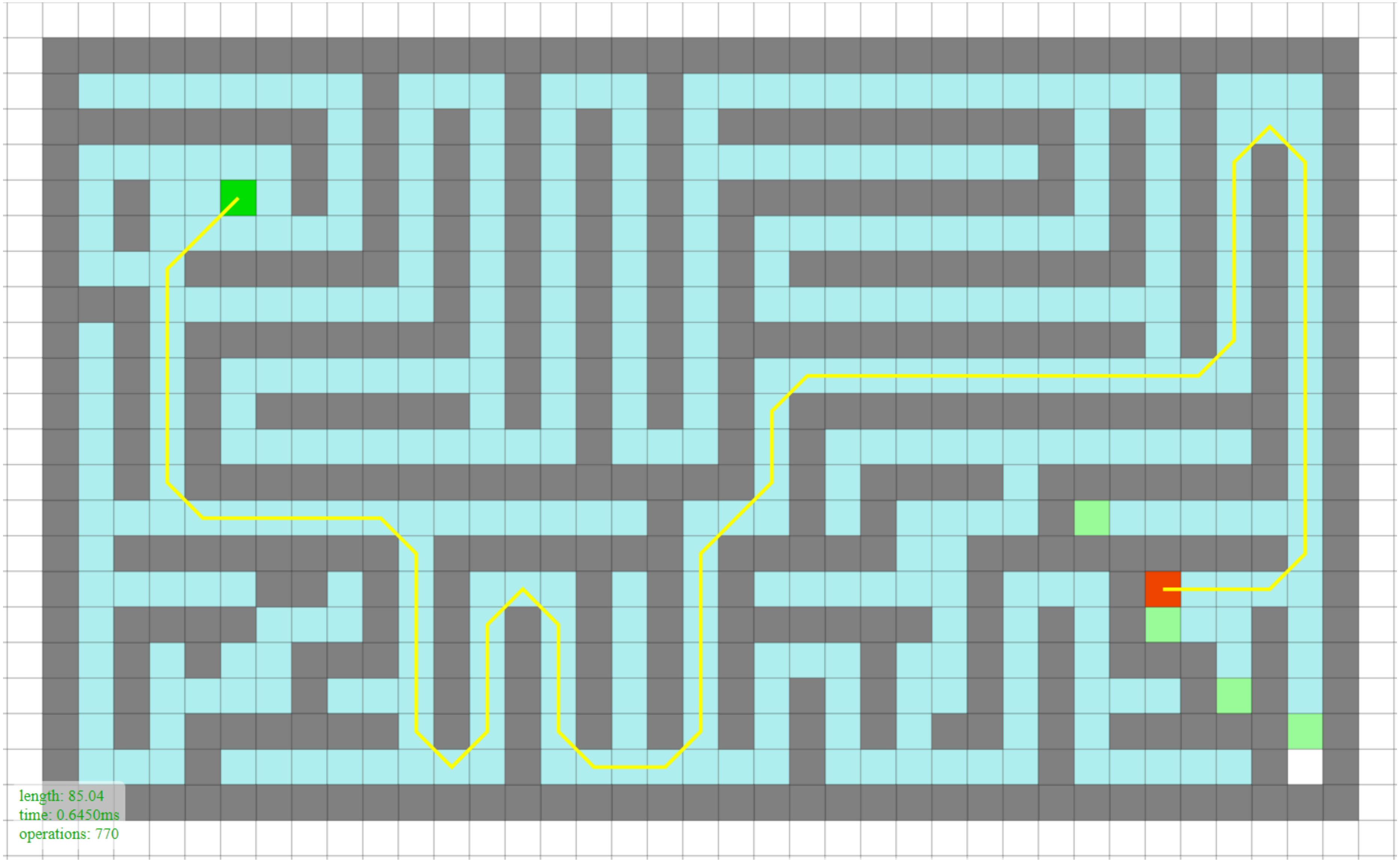




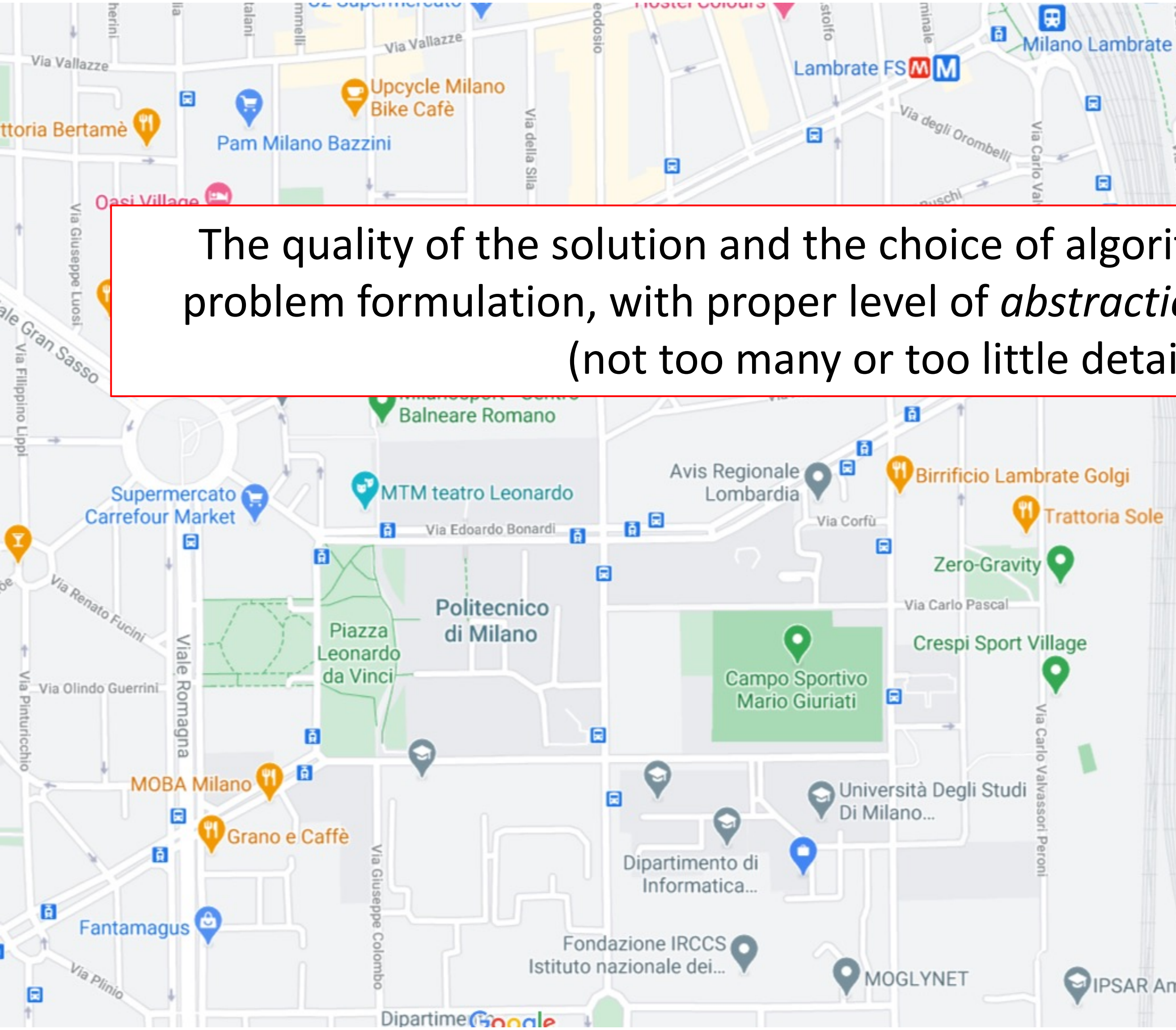
A solution



And here? Changing a few tiles, different solution

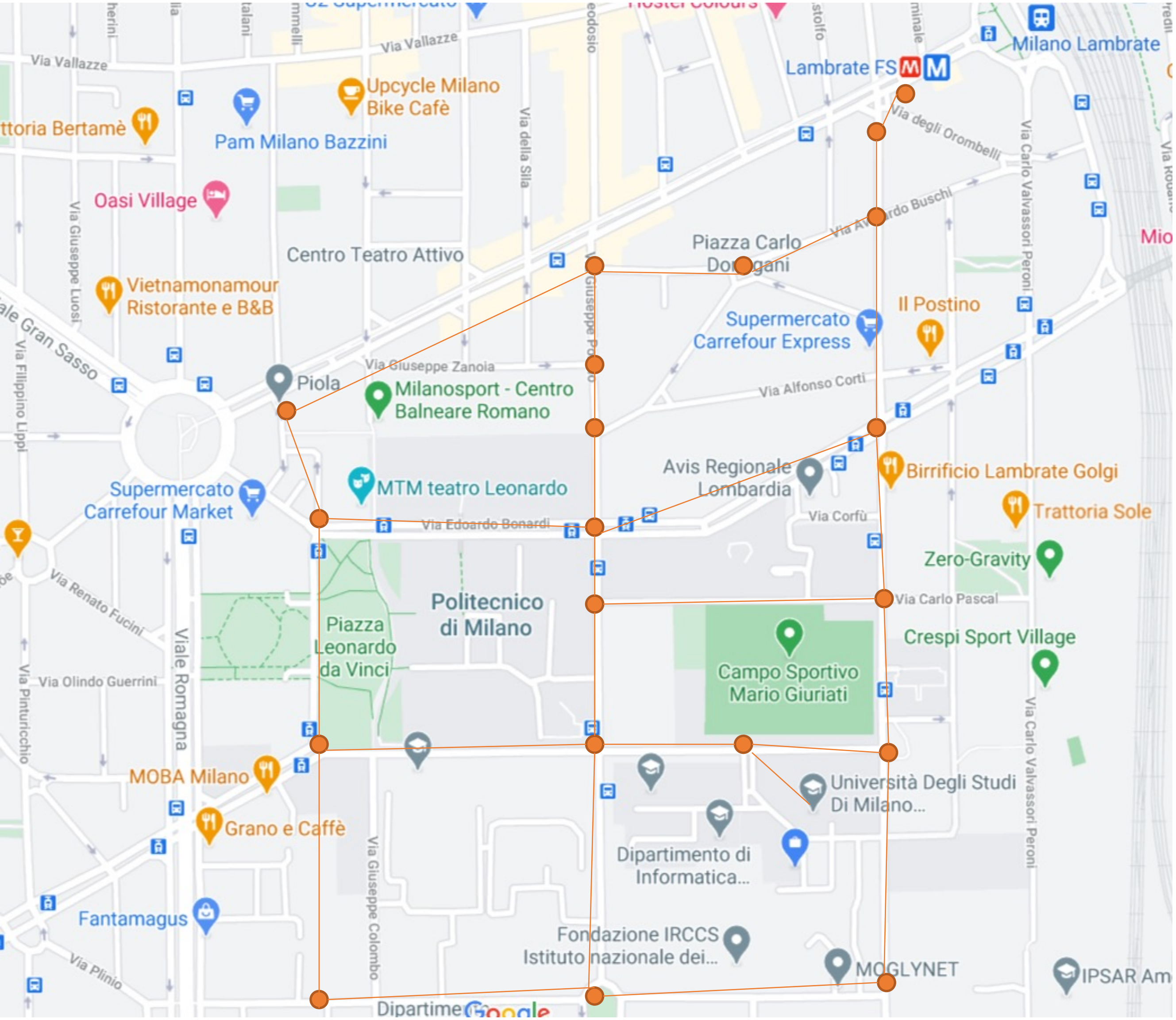


One problem, many possible ways of representing it



The quality of the solution and the choice of algorithms rely on a proper problem formulation, with proper level of *abstraction* needed for the task (not too many or too little details)

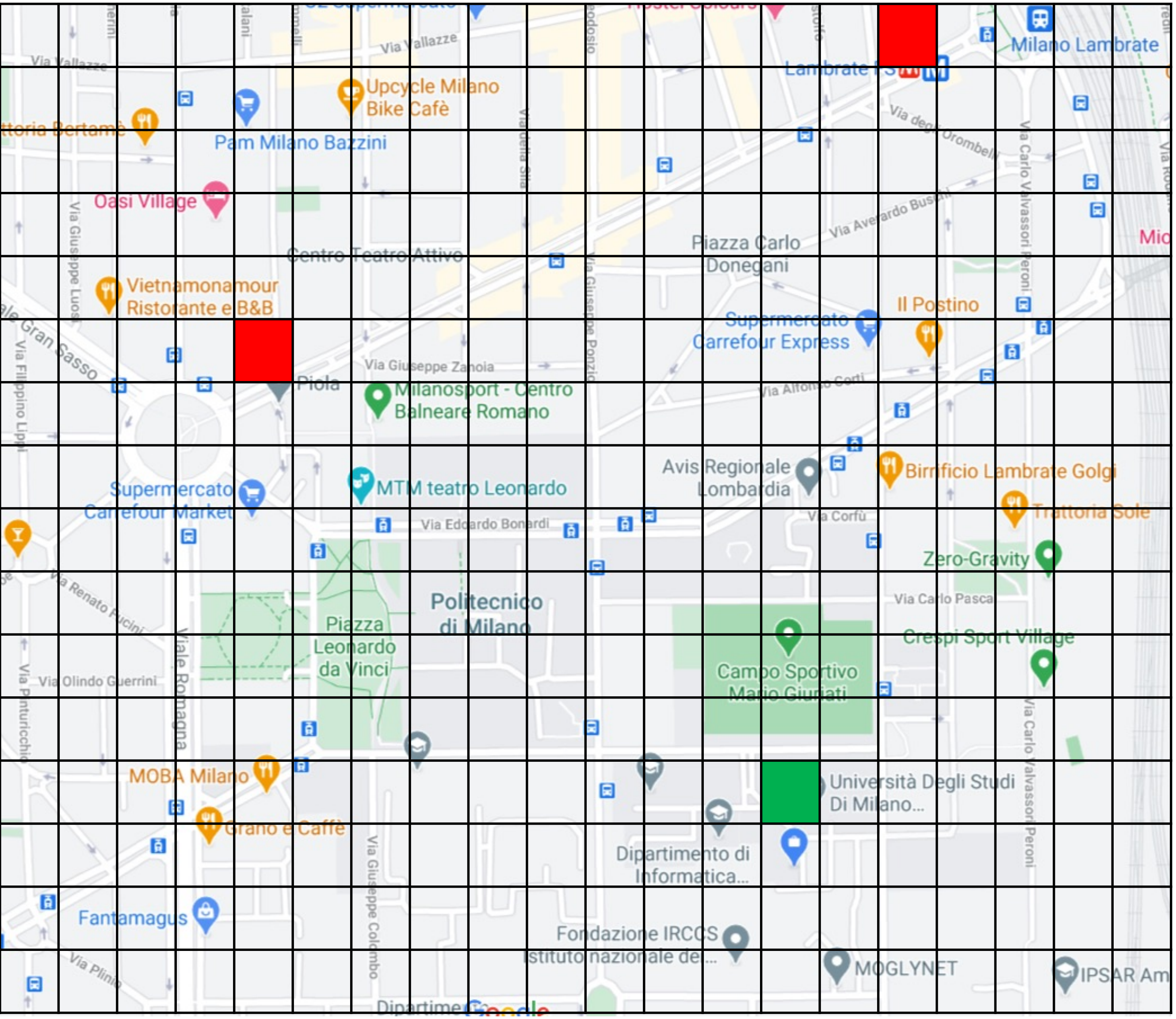
One problem, many possible ways of representing it



What type of representation?

- With which granularity?
- Shall I represent other nearby stations (Loreto, Udine?)
- Shall I represent also the bus stops?
- Trams?
- Main central stations?
- All Milan city map?
- Shall I represent all crossings and traffic lights?
- How about directions inside the campus?
- How about directions inside the building?

One problem, many possible ways of representing it



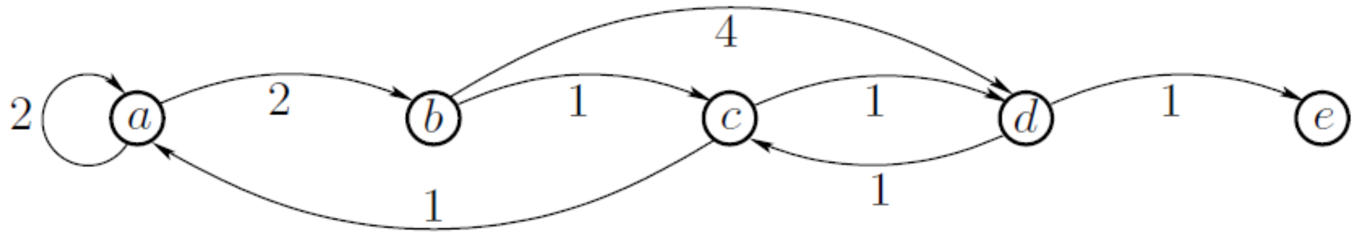
What type of representation?

- Grid map?
- How big the grid?
- Which distance?
  - Euclidean
  - Manhattan
  - ?
- Shall I represent all crossings and traffic lights?
- How about directions inside the campus? (shall I use a different grid size?)
- How about directions inside the building?

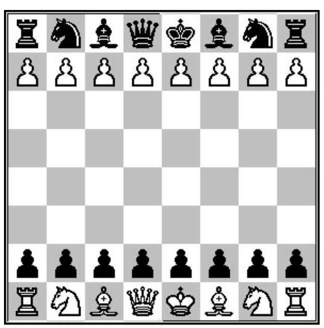
;

Problem specification

- How to **specify** a planning problem?
- First approach: provide the full state transition graph G (as in the previous example)



- Most of the times this is not an affordable option due to the combinatorial nature of the state space:



- **Chess board**: approx. 10^{47} states
- We can specify the initial state and the transition function in some compact form (e.g., set of rules to generate next states)
- The planning problem "unfolds" as search progresses

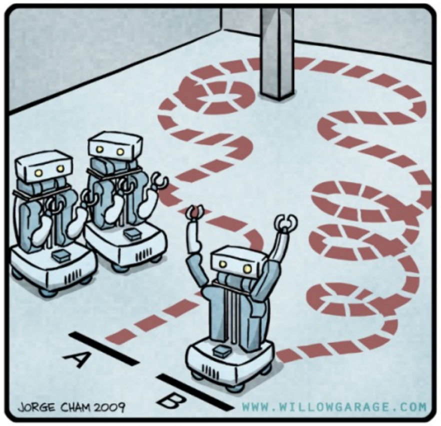
- We need an efficient procedure for *goal checking*

General features of search algorithms

A search algorithm explores the state-transition graph G until it discovers the desired solution

- feasibility: when a goal node is visited the path that led to that node is returned
- optimality: when a goal node is visited, if any other possible path to that node has higher cost the path that led to that node is returned

R.O.B.O.T. Comics



"HIS PATH-PLANNING MAY BE SUB-OPTIMAL, BUT IT'S GOT FLAIR."

Given a state and the path followed to get there, the next node to explore is chosen using a *search strategy*

It does not suffice to visit a goal node, the algorithm has to reconstruct the path it followed to get there: it must keep a trace of its search

Such a trace can be mapped to a subgraph of G, it is called *search graph*

how to evaluate a (search) algorithm?

- We can evaluate a search algorithm along different dimensions

  - **Completeness**:
    If there is a solution, is the algorithm guaranteed to find it?
    - Systematic:
      If the state space is finite, will the algorithm visit all reachable state
      (so finding a solution if a solution exists?)
  - **Optimality**:
    does the strategy find an optimal solution?
  - **Space complexity**:
    How much memory is needed to find a solution?
  - **Time complexity:**
    How long does it takes?

(The above criteria are used to evaluate a broader class of algorithms)

Soundness

- Optimality: *does the returned solution lead to a goal with minimum cost?*

Maybe we are not always looking for the optimal solution…

…for some problems, we may look for other features

Soundness: If the algorithm returns a solution, is it compliant with the desired features specified in the problem formulation?

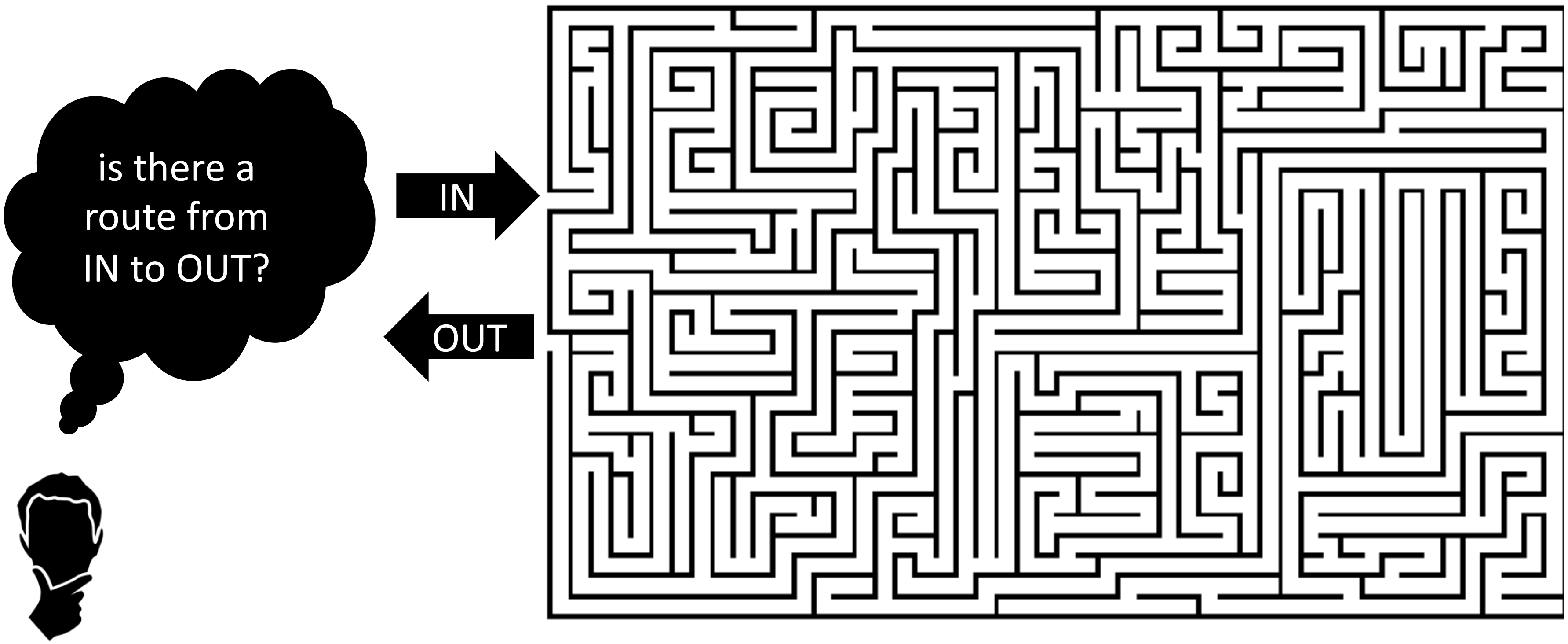
- Example:
  - Feasibility: *does the returned solution lead to a goal?*
  - Optimality: *does the returned solution lead to a goal with minimum cost?*

  (We may need other features from the algorithm e.g., approximation)

Completeness and the systematic property

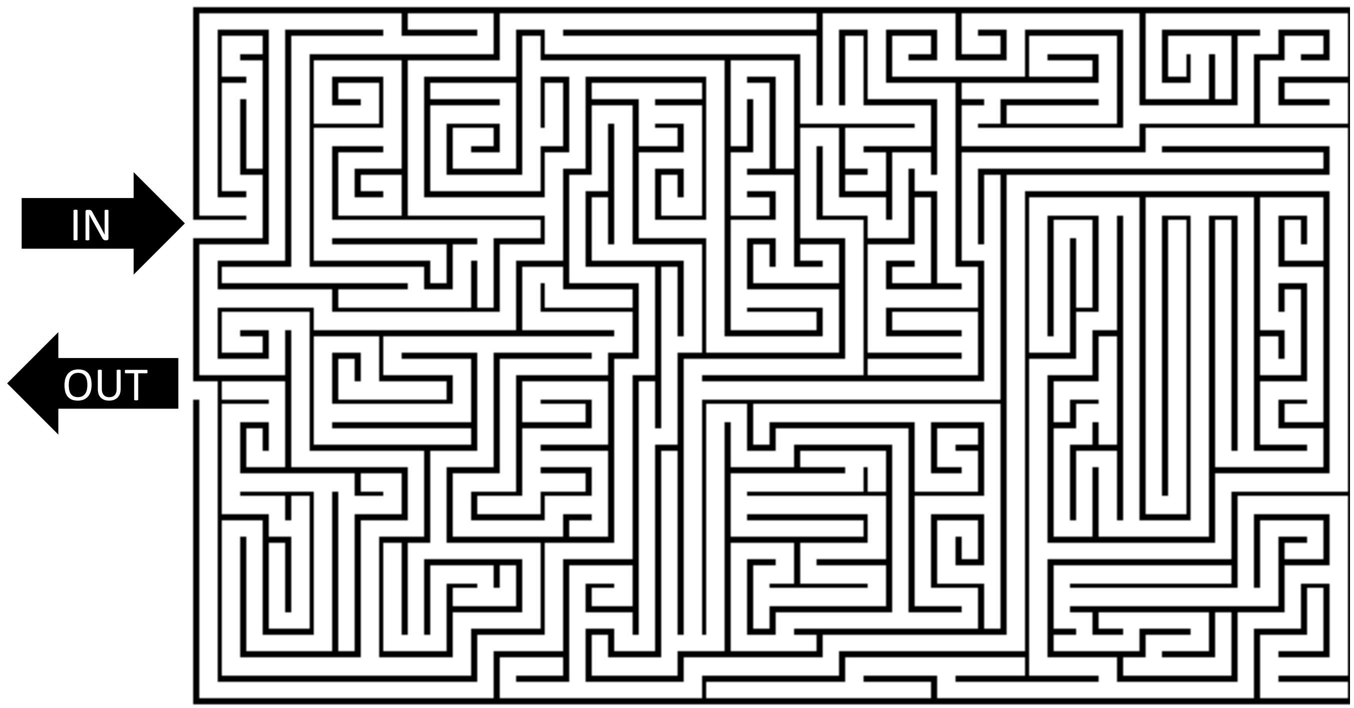
- If a solution exists, does the algorithm find it?
- Typically shown by proving that the search will/will not visit all states if given enough time → systematic
- If the state-space is finite, ensuring that no redundant exploration occurs is sufficient to make the search systematic.
- If the state space is infinite, we can ask if the search is systematic:
  - If there is a solution, the search algorithm must report it in finite time
  - if the answer is no solution, it’s ok if it does not terminate but …
  - … all reachable states must be visited in the limit: as time goes to infinity, all states are visited – all reachable vertex is explored - (this definition is sound under the assumption of countable state space)

Visual example



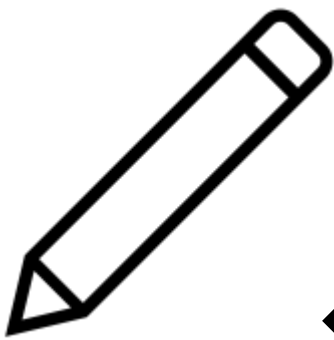
Visual example

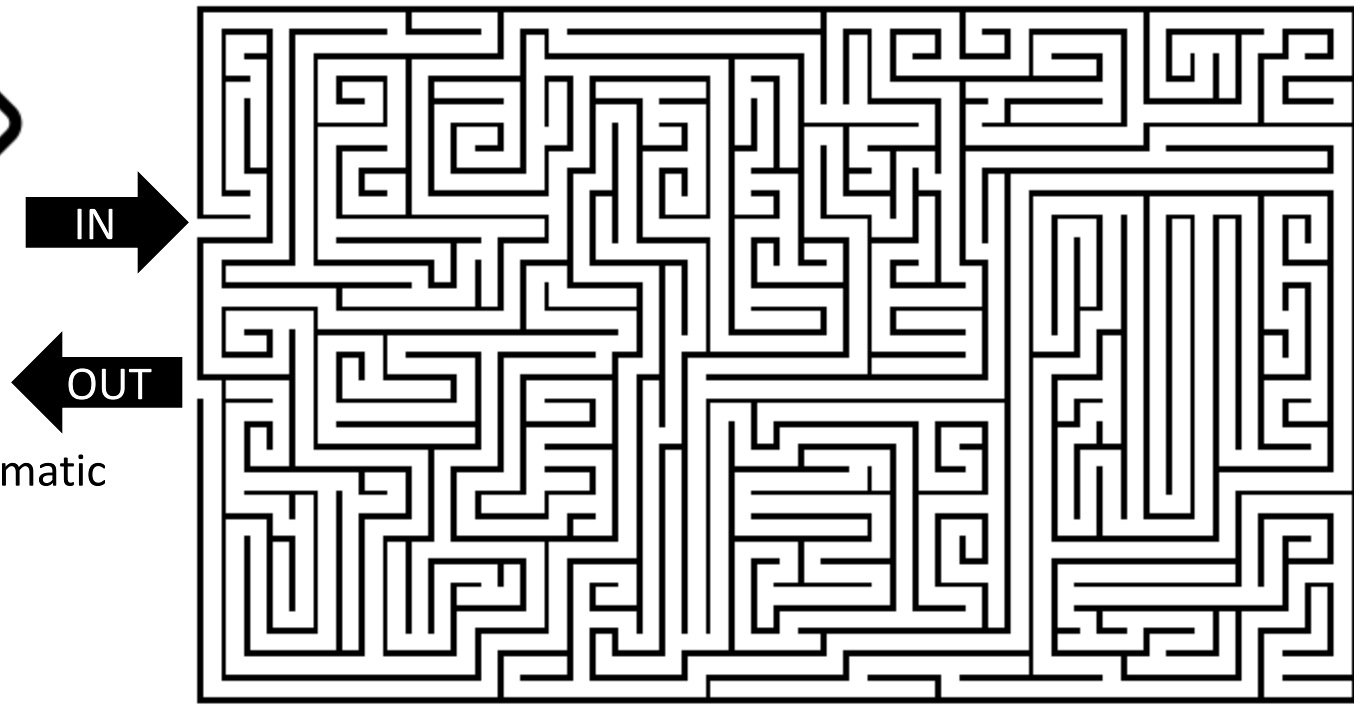
Complete / Systematic



- Searching along **multiple** trajectories (either concurrently or not), eventually covers all the reachable space

Visual example



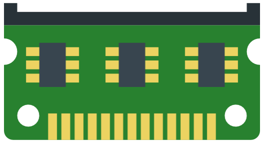


Not complete / Not systematic

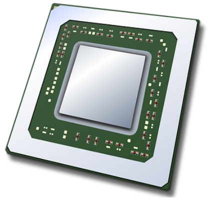
- Searching along a **single** trajectory, eventually gets stuck in a dead end (or find a solution if we are lucky)

Space and time complexity

- Space complexity: how does the amount of memory required by the search algorithm grows as a function of the problem's dimension (worst case)?



- Time complexity: how does the time required by the search algorithm grows as a function of the problem's dimension (worst case)?



- Asymptotic trend:
  - We measure complexity with a function $f(n)$ of the input size
  - For analysis purposes, the "Big O" notation is convenient:

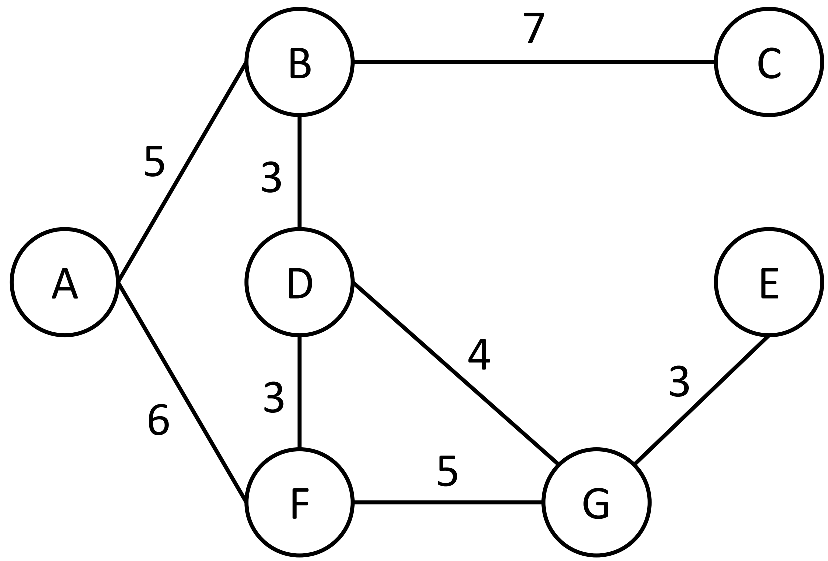
$$\text{A function } f(n) \text{ is } O(g(n)) \text{ if } \exists k > 0, n_0 \text{ such that } f(n) \leq kg(n) \text{ for } n > n_0$$

- An algorithm that is $O(n^2)$ is better than one that is $O(n^5)$
- If $g(n)$ is an exponential, the algorithm is not efficient

Running example

- To present the various search algorithms, we will use this *problem instance* as our running example

State-transition graph:



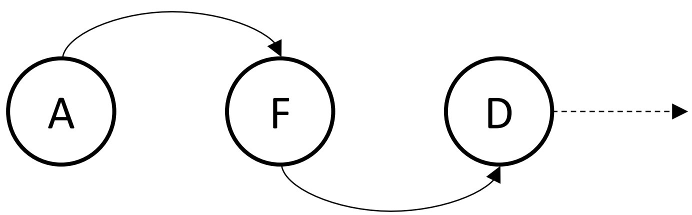
Initial state: A

Desired solution: any path to goal state E

- It might be useful to think it as a map, but keep in mind that this interpretation does not hold for every instance

Search algorithm definition

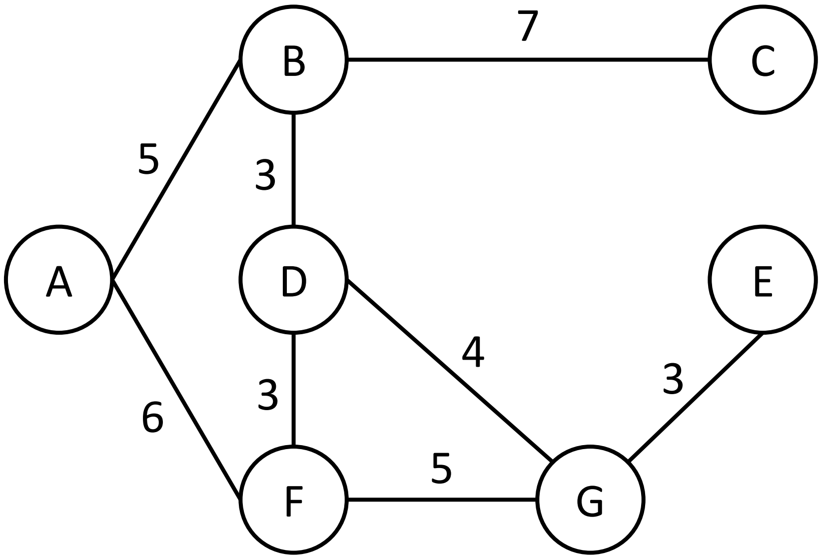
- The different search algorithms are substantially characterized by the answer they provide to the following question:



Given what I searched so far,
where to search next?
(search strategy)

- The answer is encoded in a set of rules that drives the search and define its type, let’s start with the simplest one

Depth-First Search (DFS)



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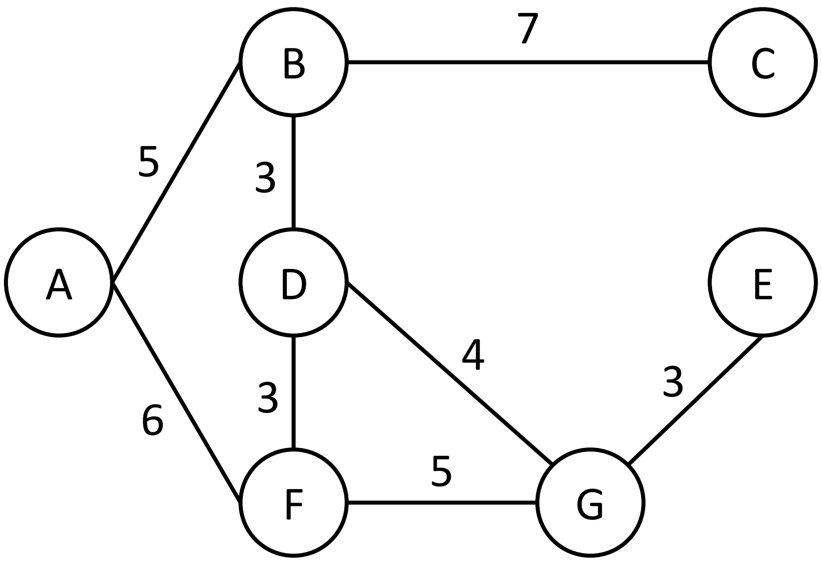
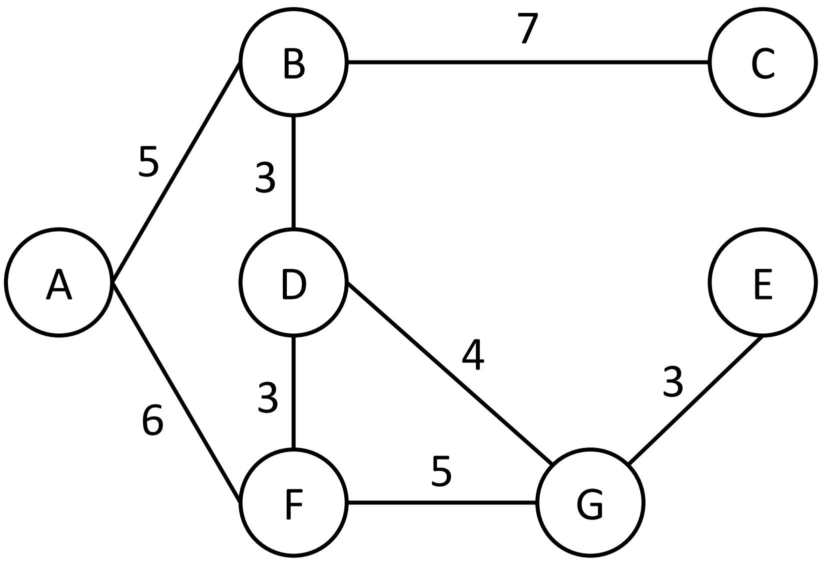
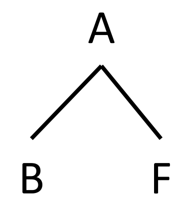


A

Depth-First Search (DFS)





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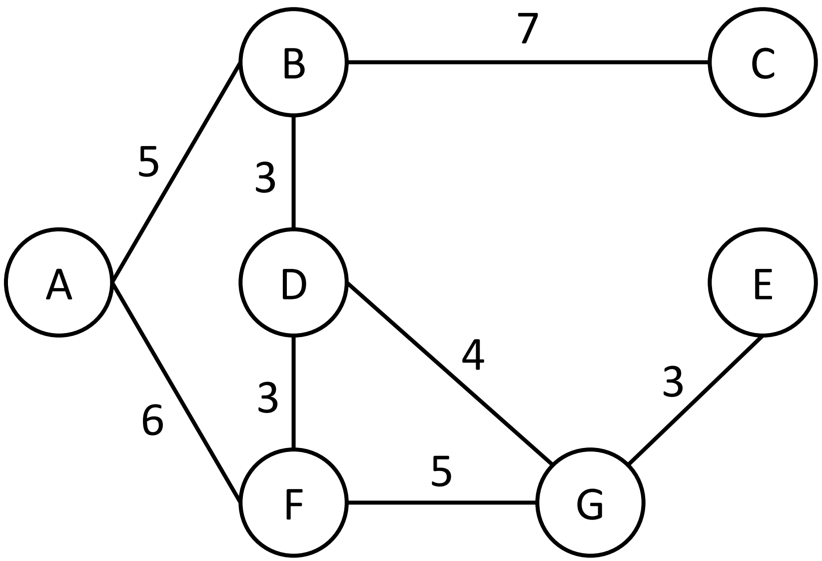

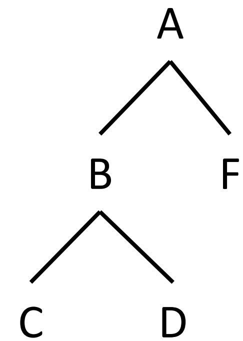

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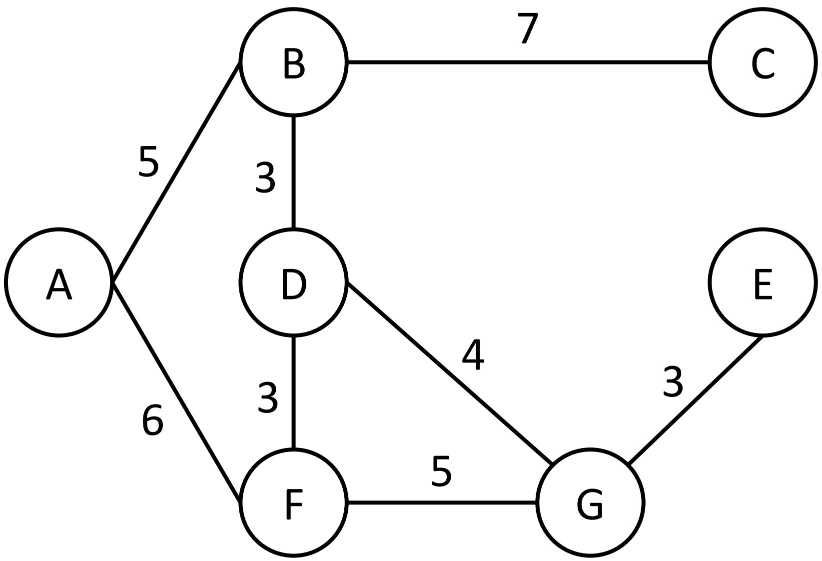

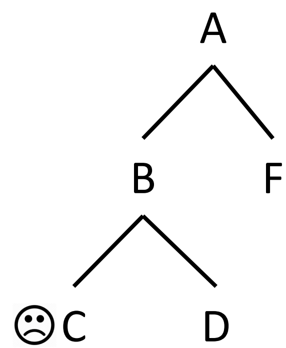

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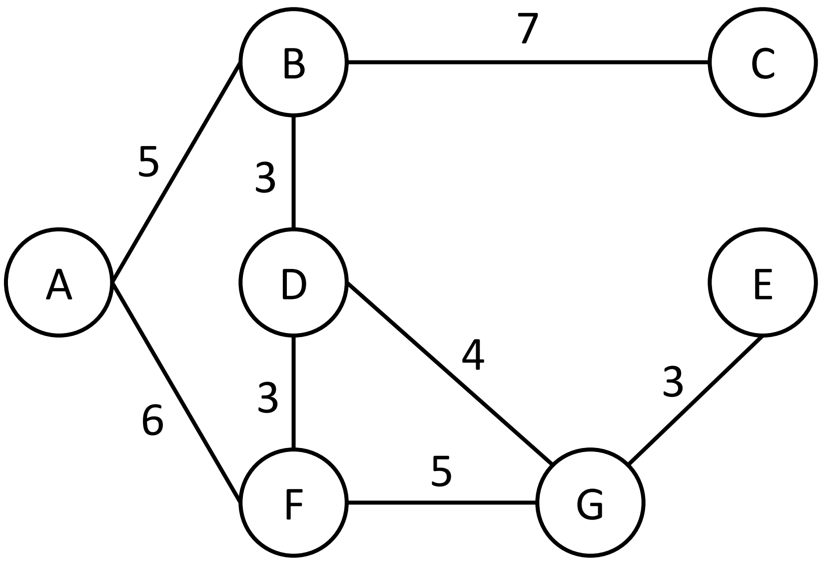

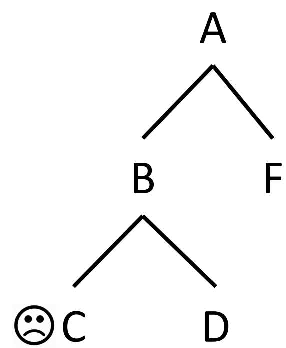

- A Depth-First Search (DFS) chooses the deepest node in the search tree (How to break ties? For now, lexicographic order)

- A dead end stopped the search, DFS seems not complete. Can we fix this?

- Let's endow our DFS with **backtracking**: a way to reconsider previously evaluated decisions

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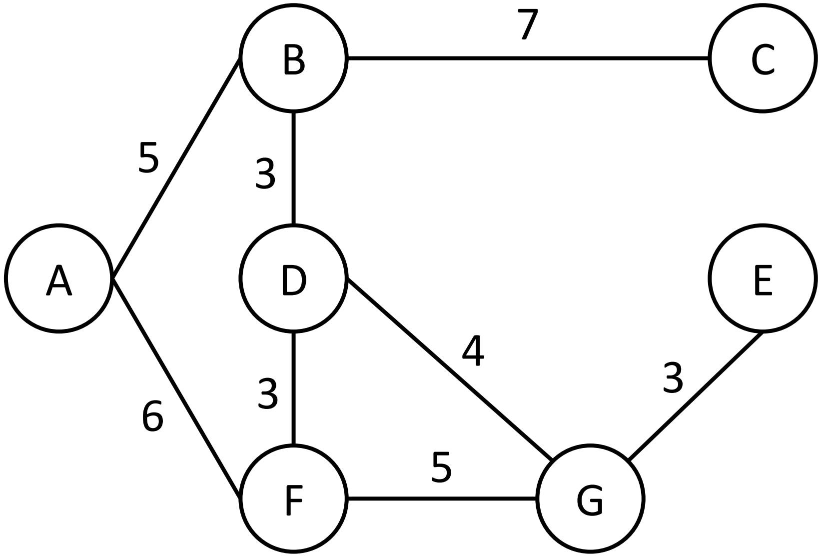

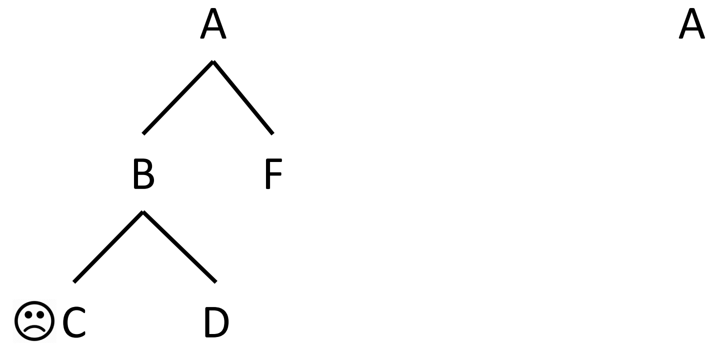

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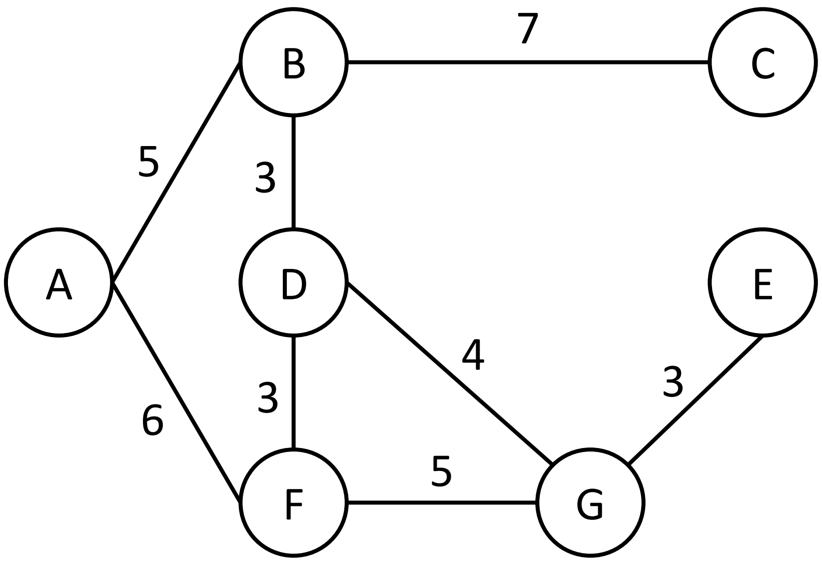

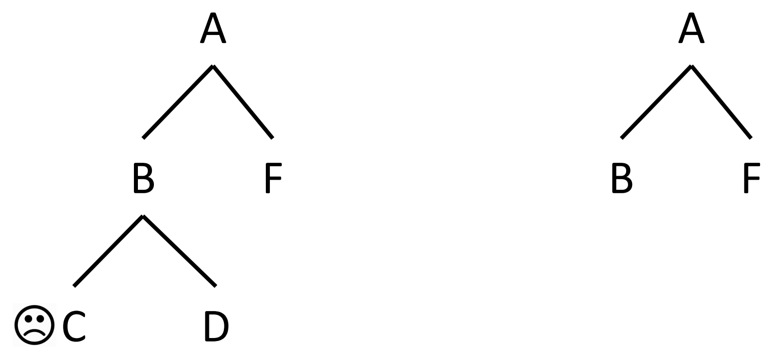

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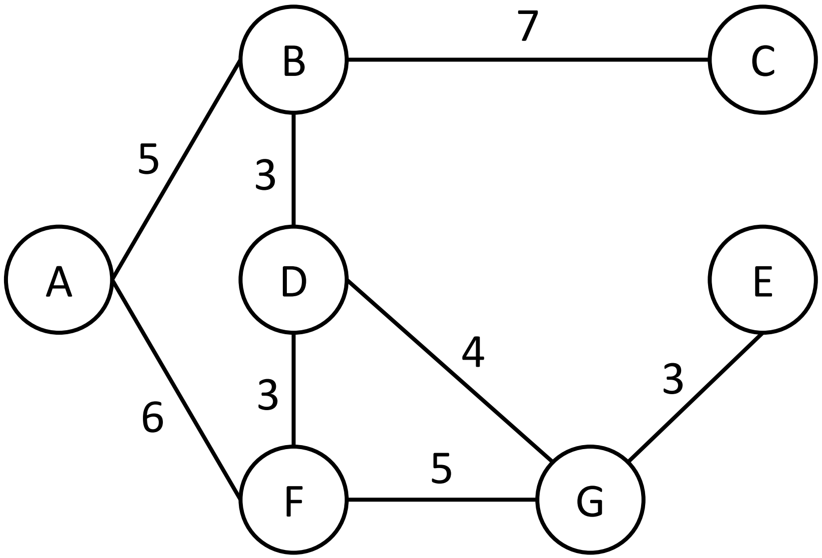

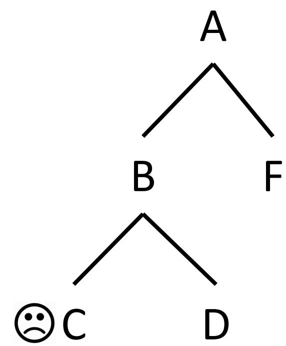

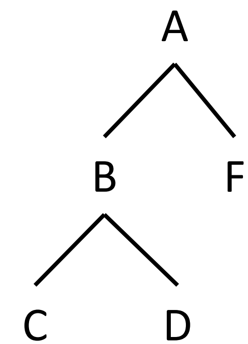

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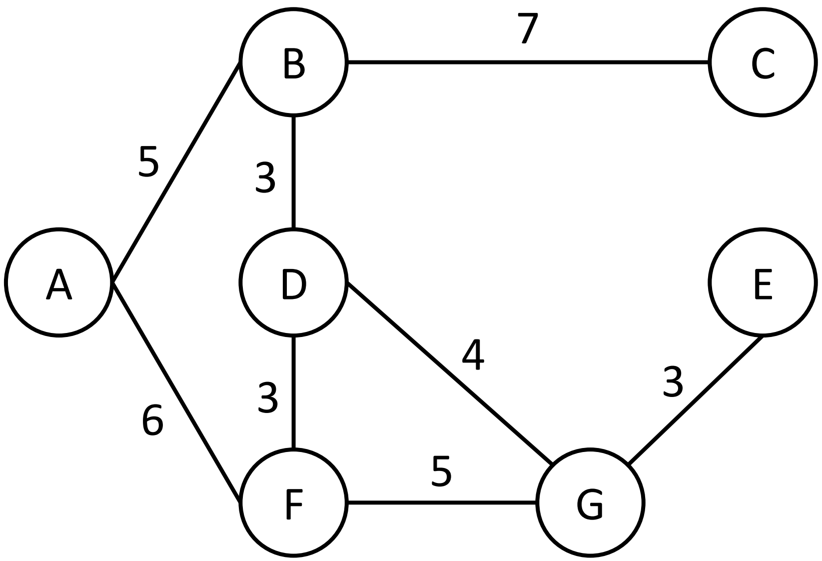
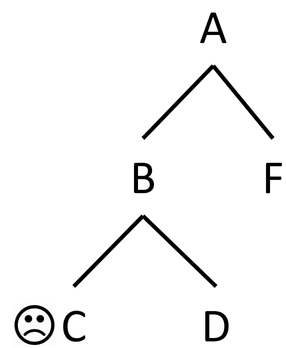
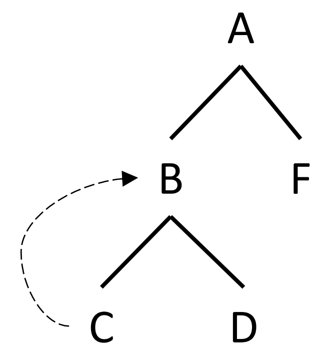
Depth-First Search (DFS)







- A Depth-First Search (DFS) chooses the deepest node in the search tree (How to break ties? For now lexicographic order)

- A dead end stopped the search, DFS seems not complete. Can we fix this?

- Let’s endow our DFS with **backtracking**: a way to reconsider previously evaluated decisions

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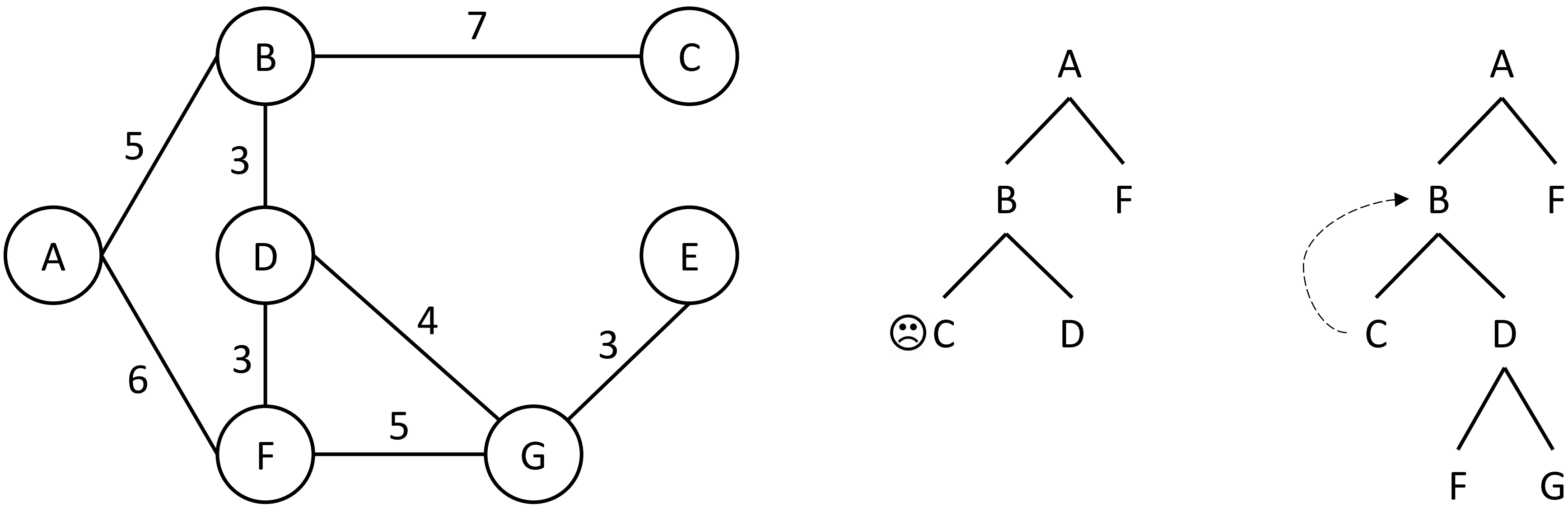

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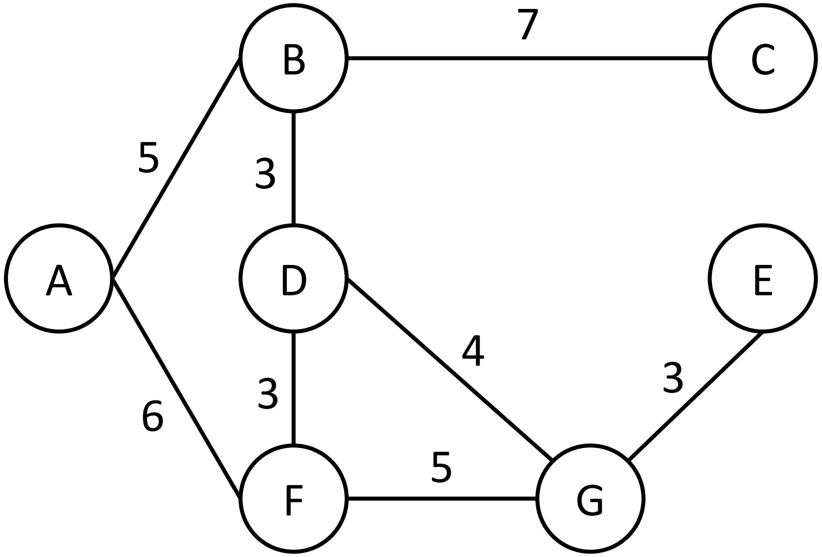

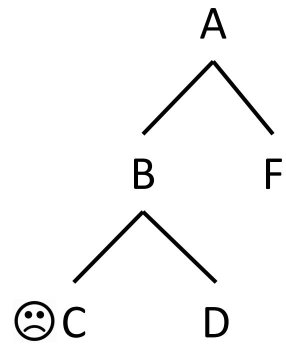

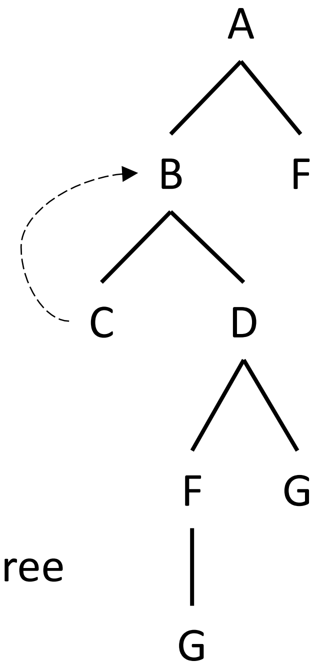

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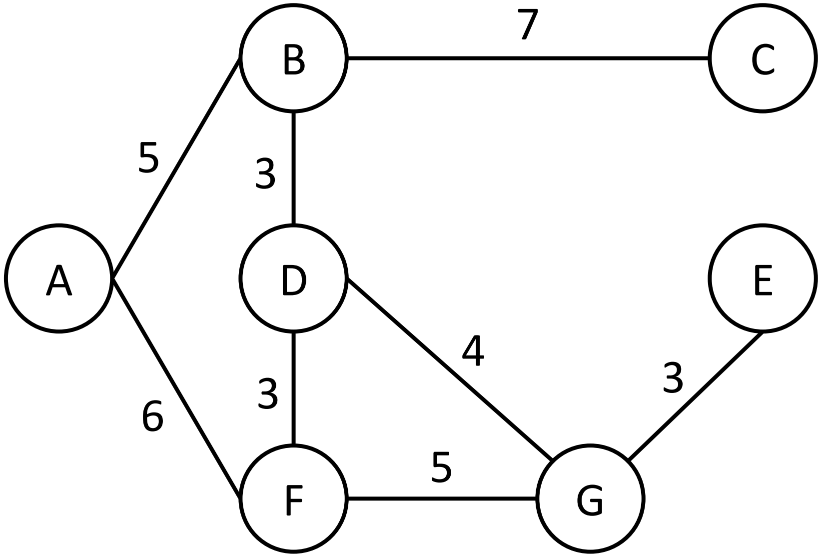

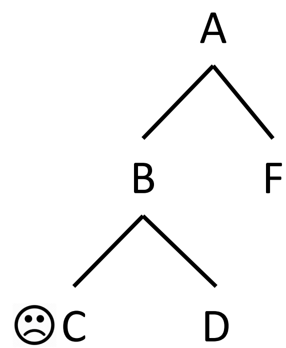

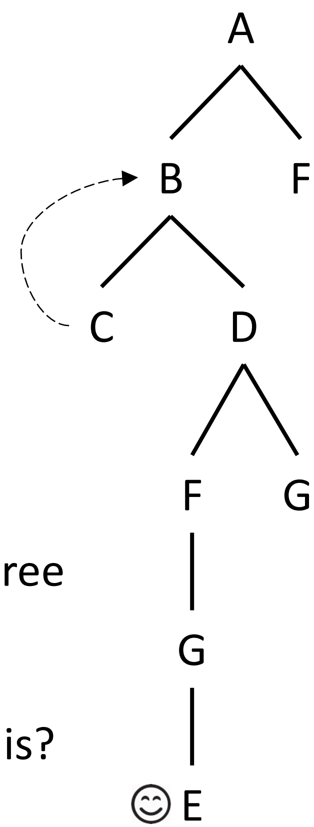

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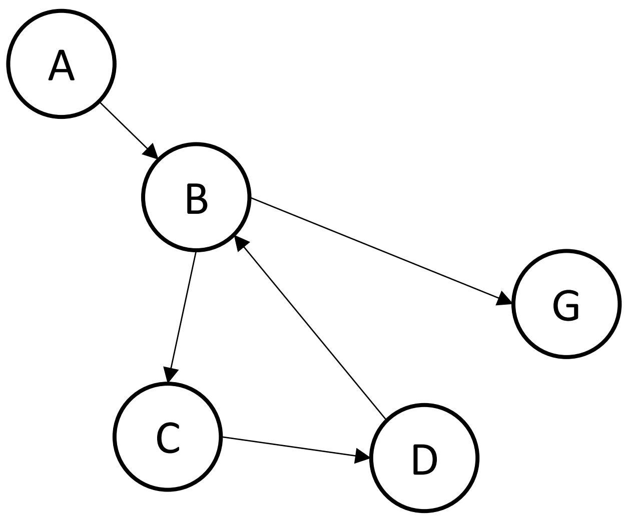
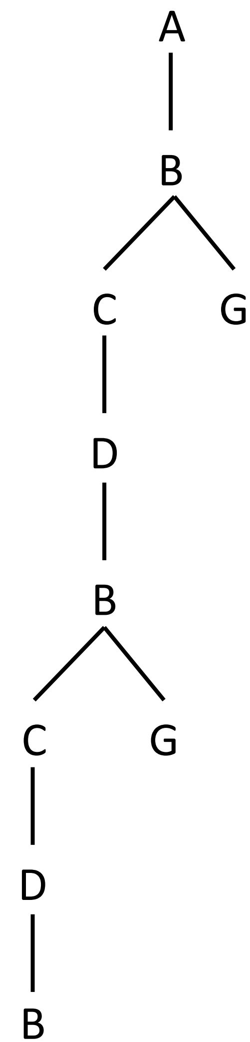
Solution: (A->B->D->F->G->E)

Depth-First Search (DFS) and Loops



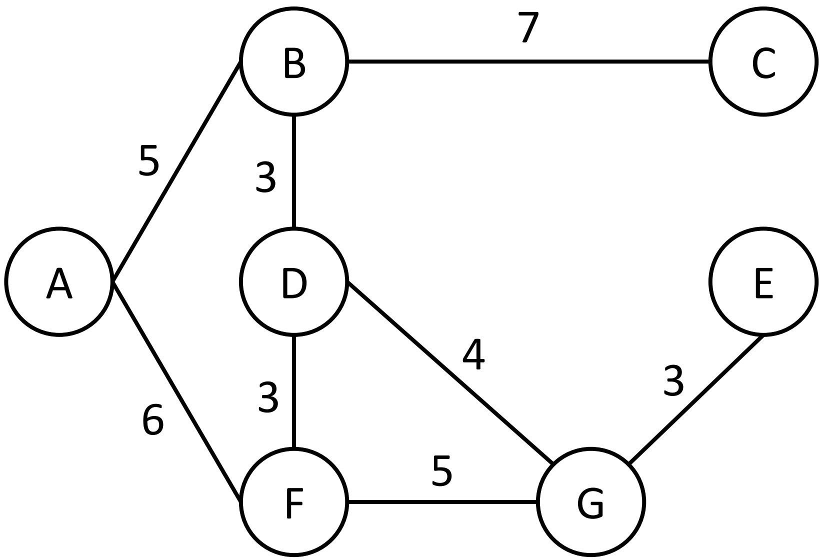


- DFS with loops –> non systematic / complete
- We want to **avoid loops** on the same branch (loops are redundant paths)

Depth-First Search (DFS)

- DFS with loops removal and BT is sound and complete (for finite spaces)
- Call b the maximum branching factor, i.e., the maximum number of actions available in a state
- Call d the maximum depth of a solution, i.e., the maximum number of actions in a path
- Space complexity: $O(d)$
- Time complexity: $1 + b + b^2 + \ldots + b^d = O(b^d)$

Breadth-First Search (BFS)



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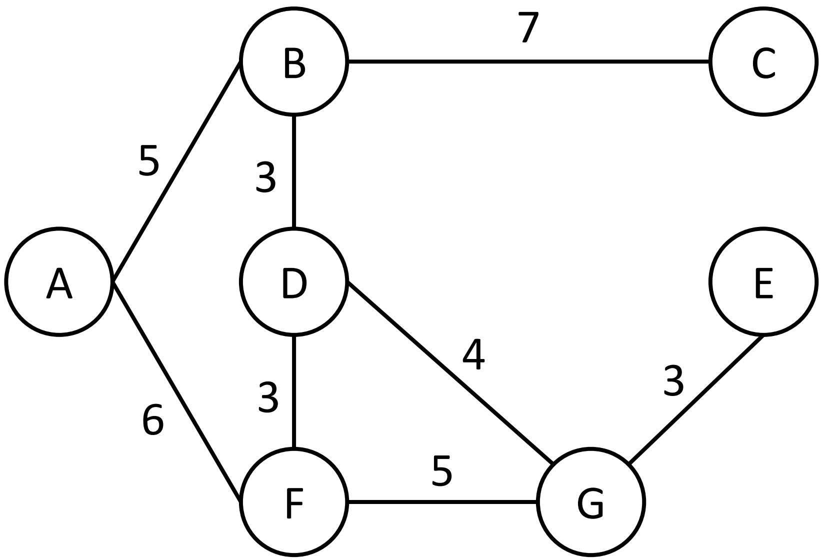

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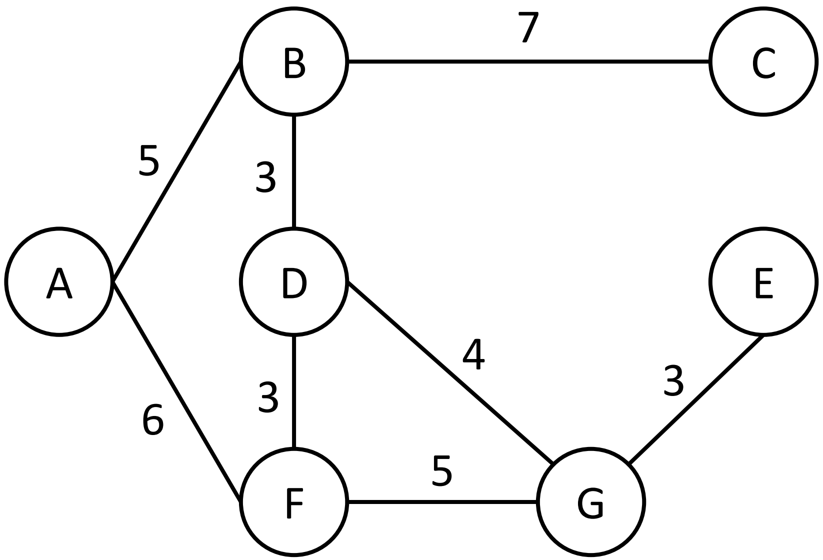

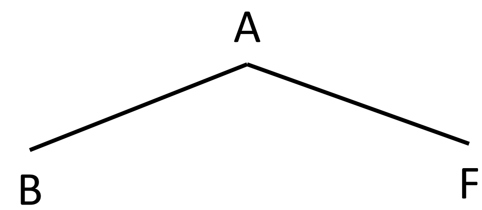

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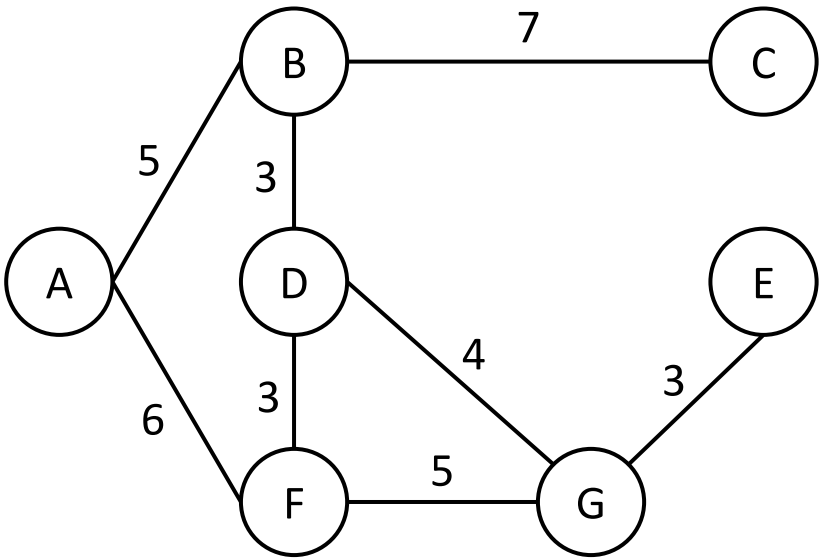

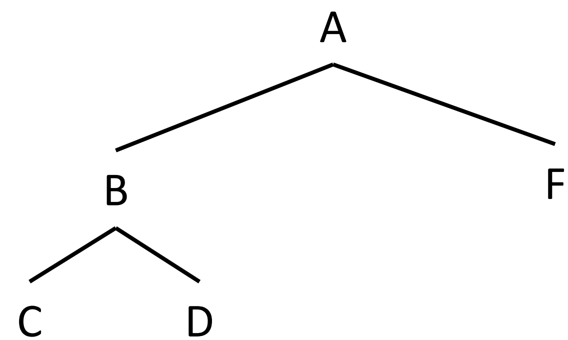

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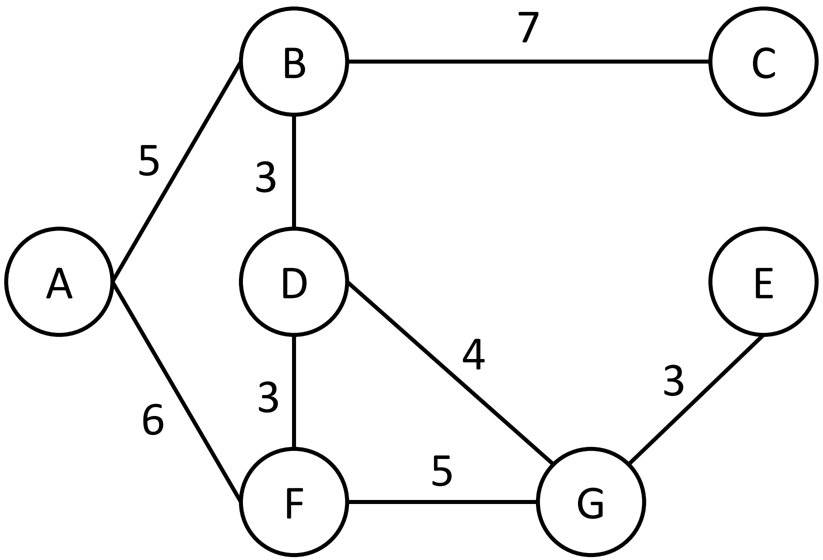

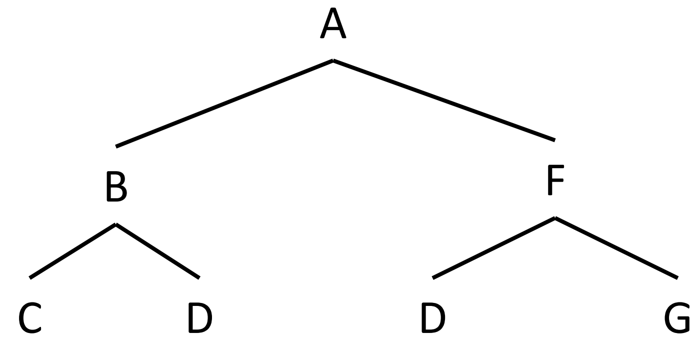

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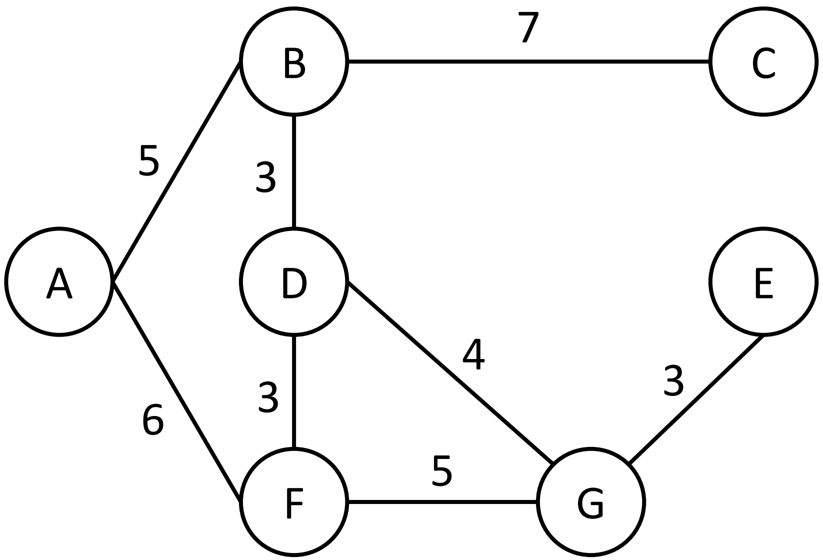

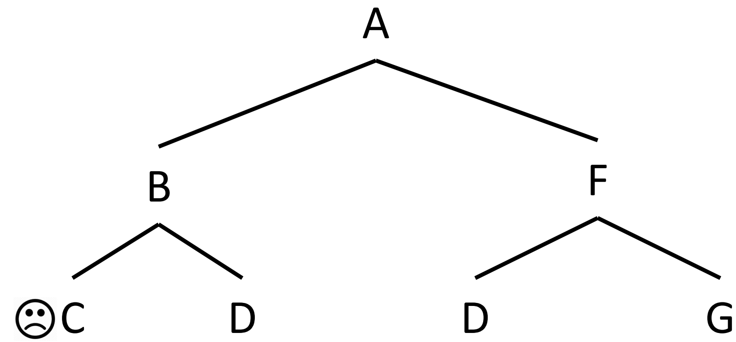

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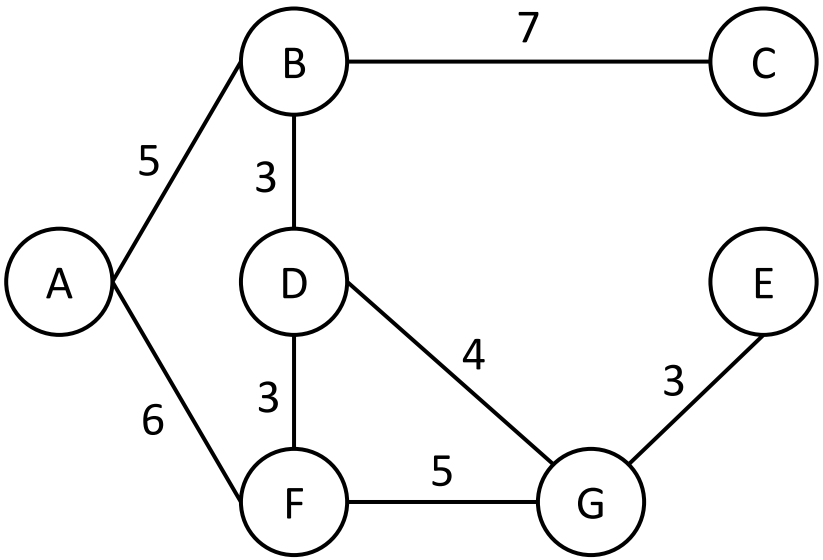

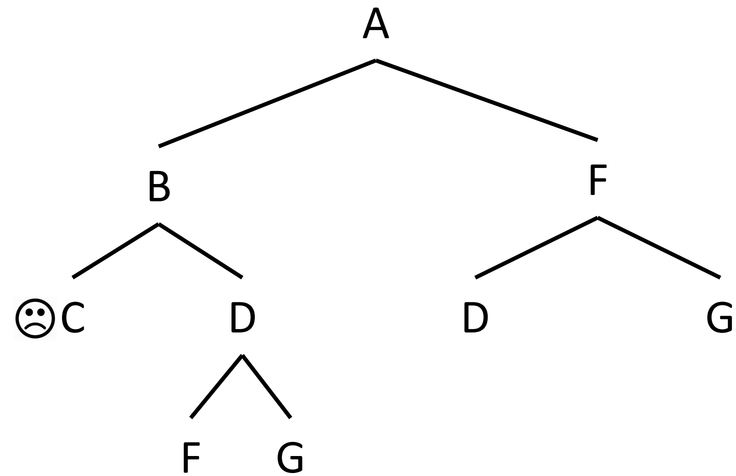

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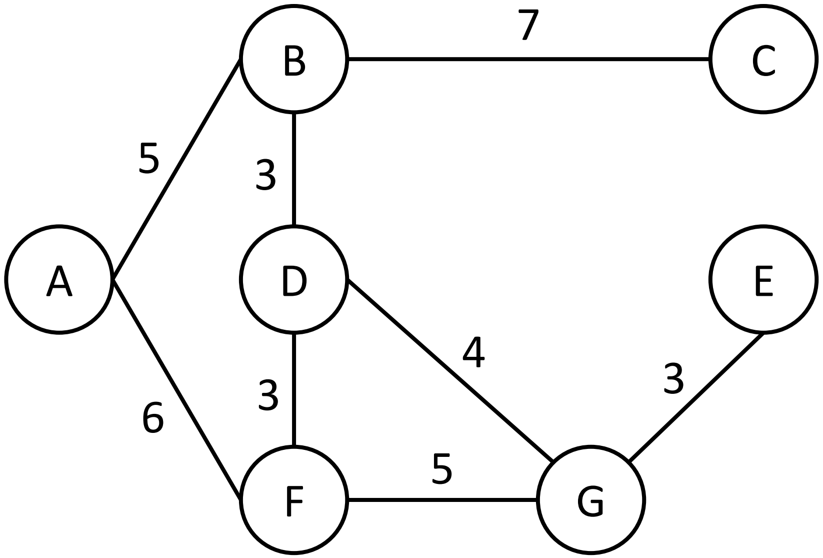

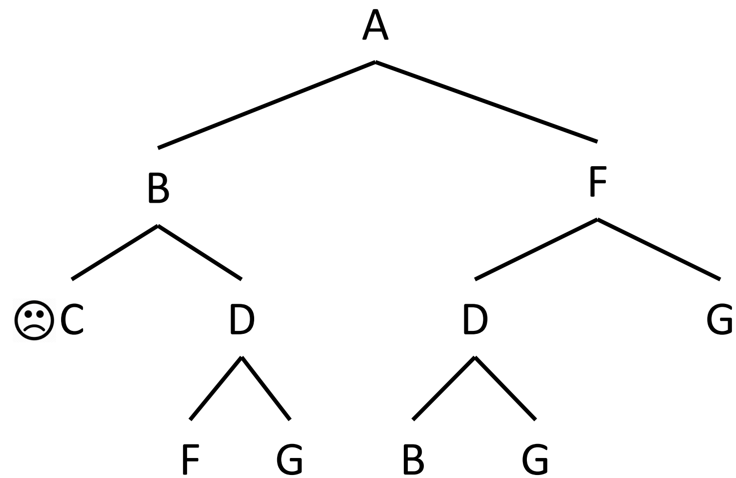

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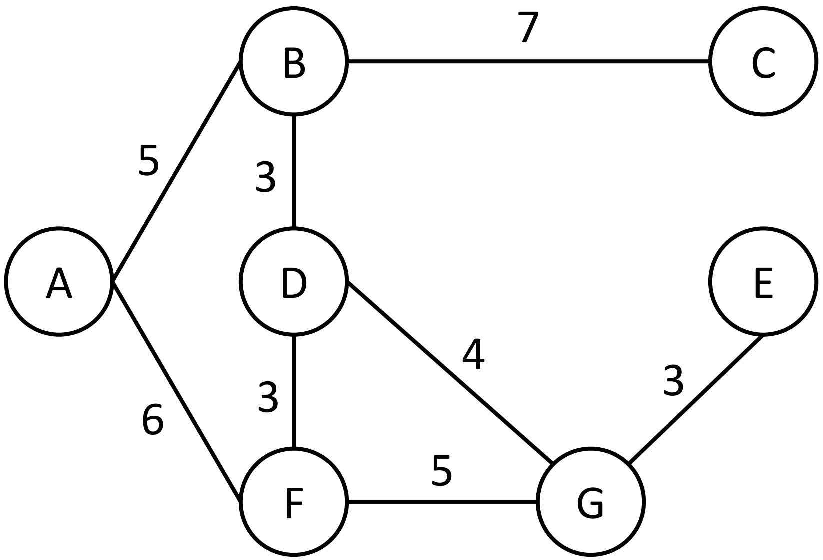
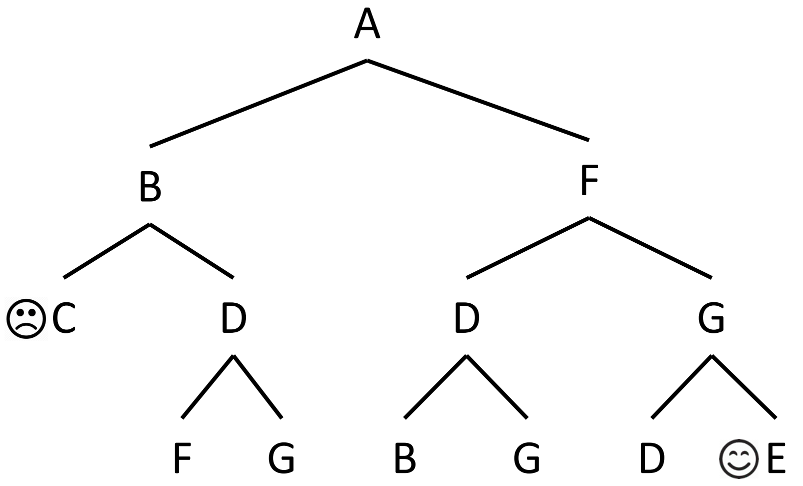
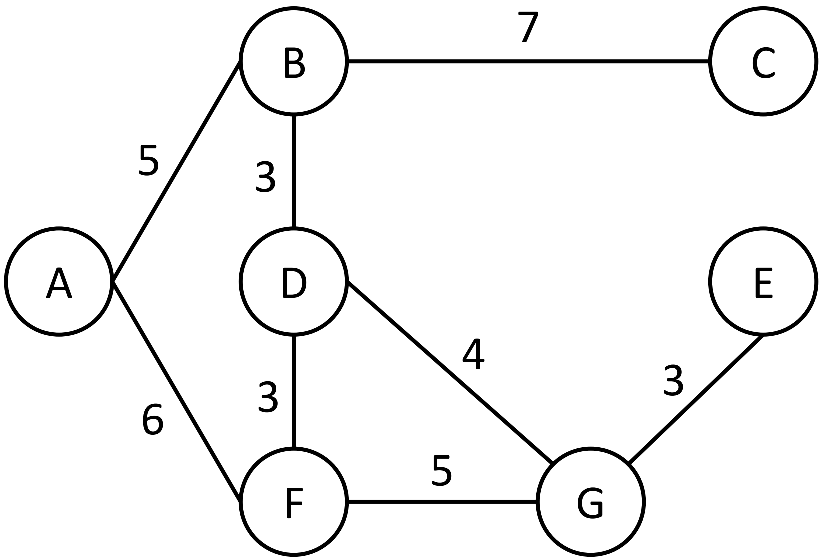
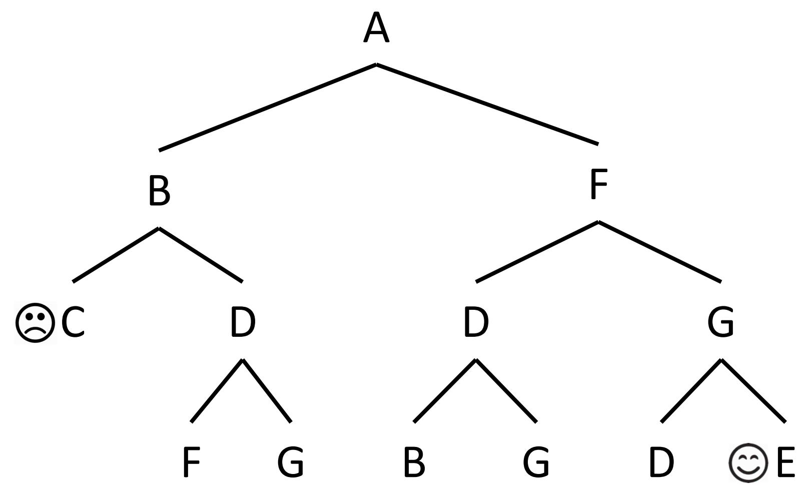




Solution: (A->F->G->E)

Breadth-First Search (BFS)





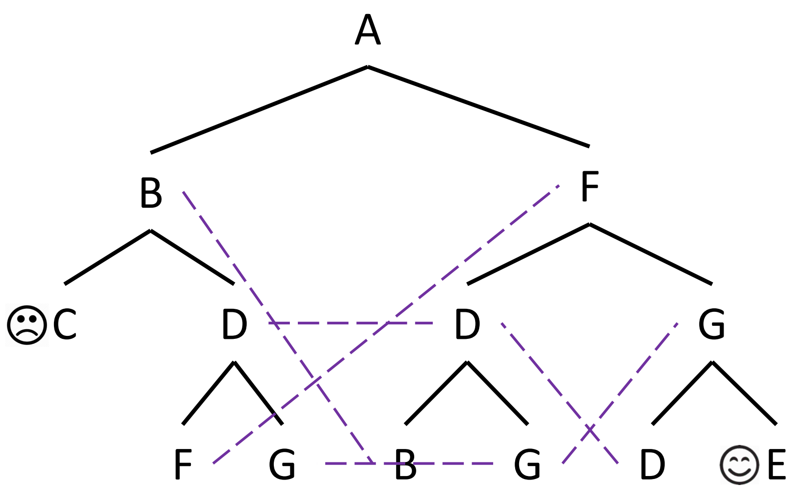
Solution: (A->F->G->E)

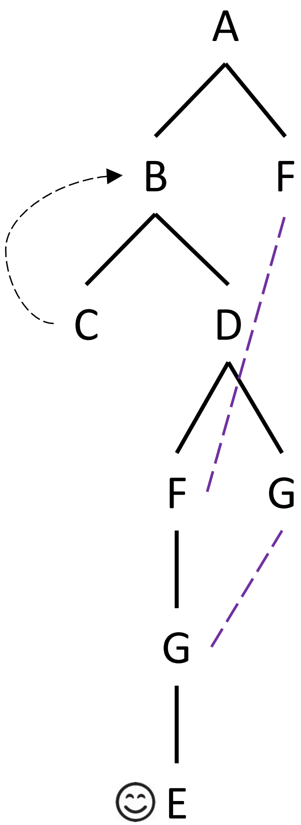
- A Breadth-First Search (BFS) chooses the shallowest node, thus exploring in a level-by-level fashion
- It has a more conservative behavior and does not need to reconsider decisions
- Call q the depth of the shallowest solution (in general $q \leq d$)
- Space complexity: $O(b^q)$
- Time complexity: $O(b^q)$

Redundant paths

- Both DFS and BFS visited some nodes multiple times (avoiding loops prevents this to happen only within the same branch)

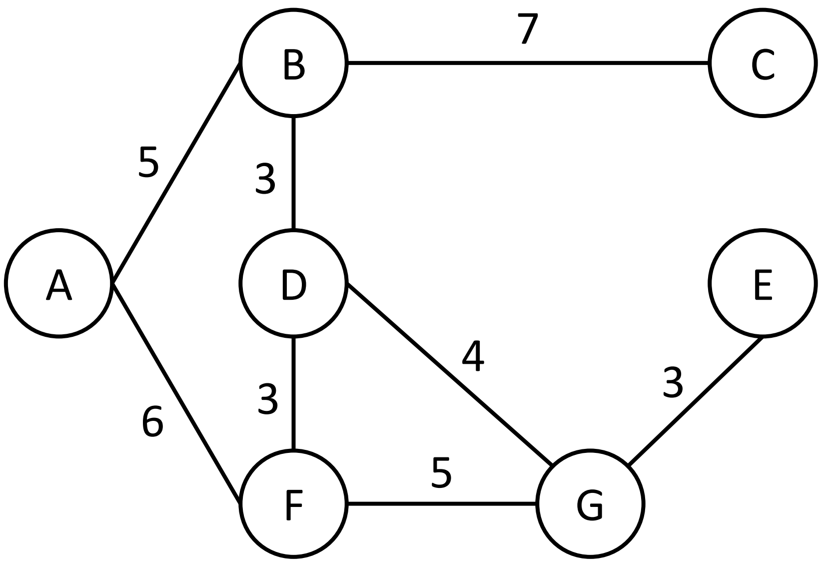
- In general, this does not seem very efficient. Why?





- Idea: discard a newly generated node if already present somewhere on the tree, we can do this with an **enqueued list**

DFS with Enqueued List



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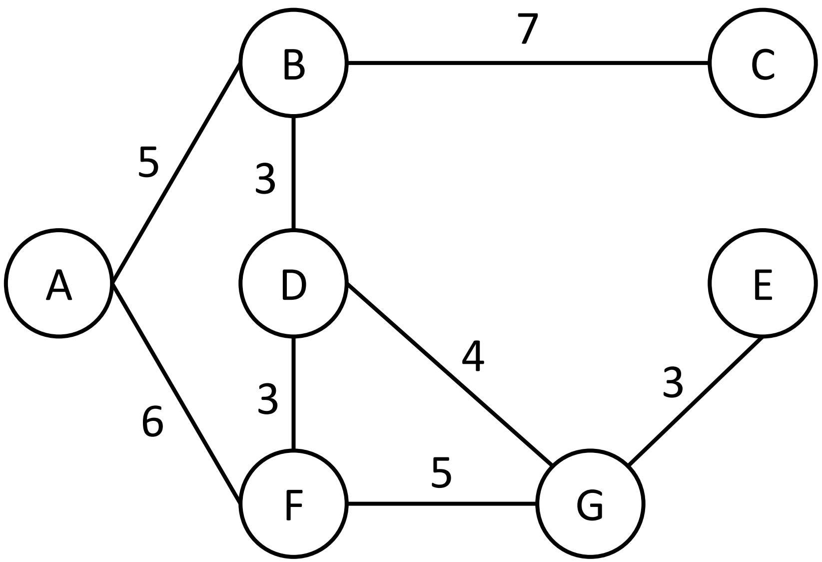

A

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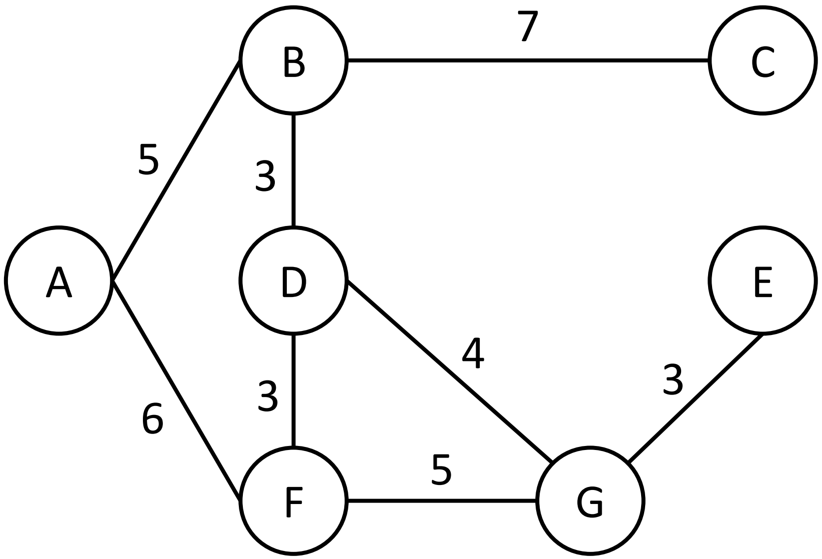

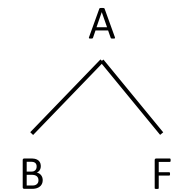

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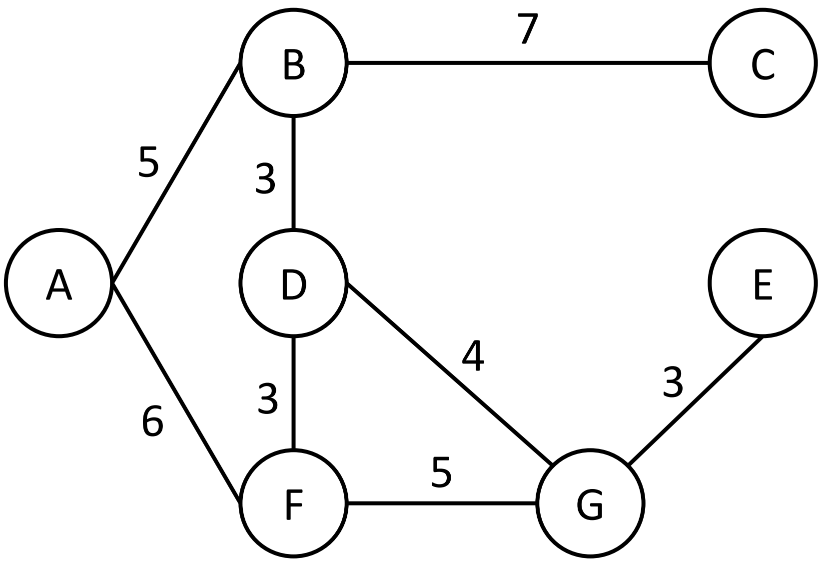

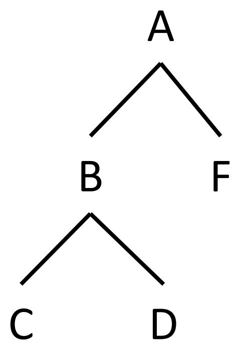

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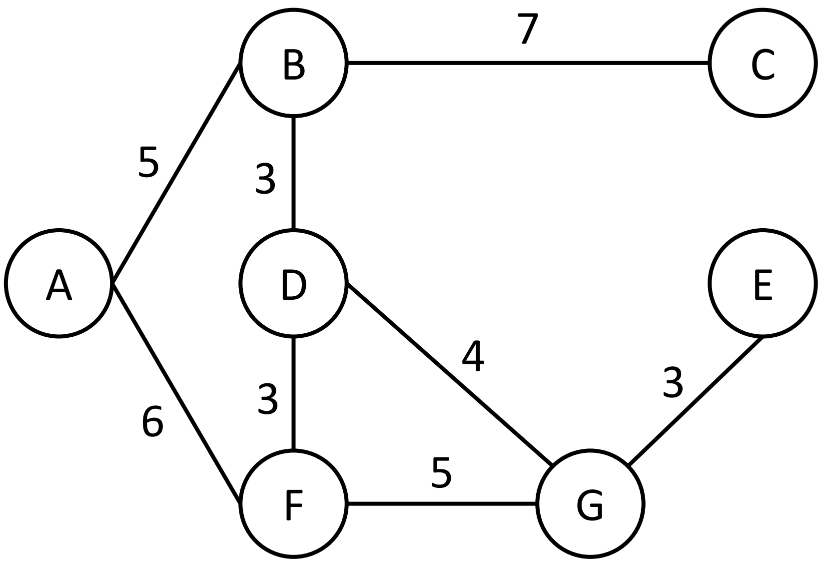

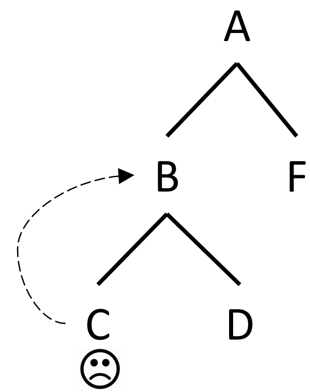

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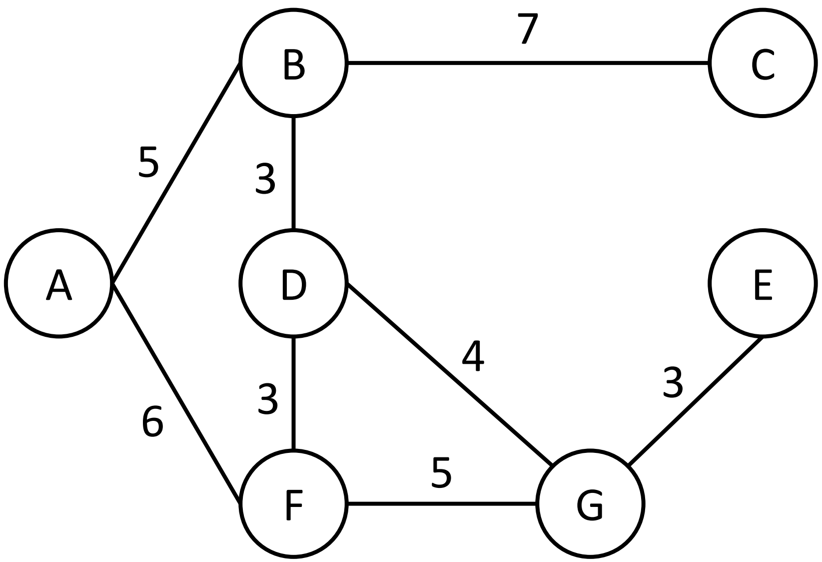
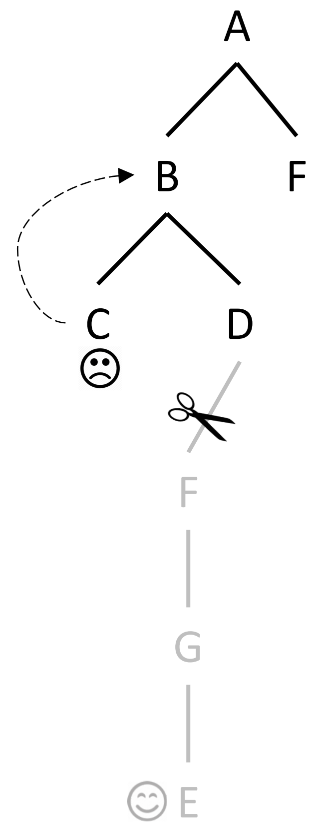
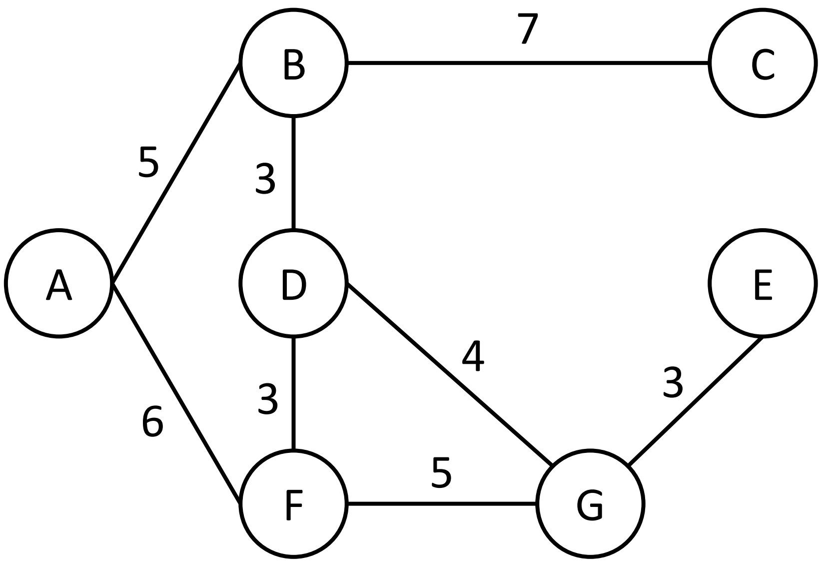
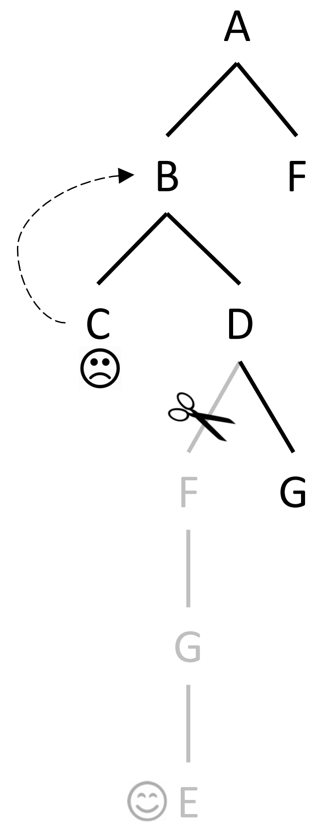




- Node F ha already been “enqueued” on the tree, by discarding it we *prune* a branch of the tree

DFS with Enqueued List





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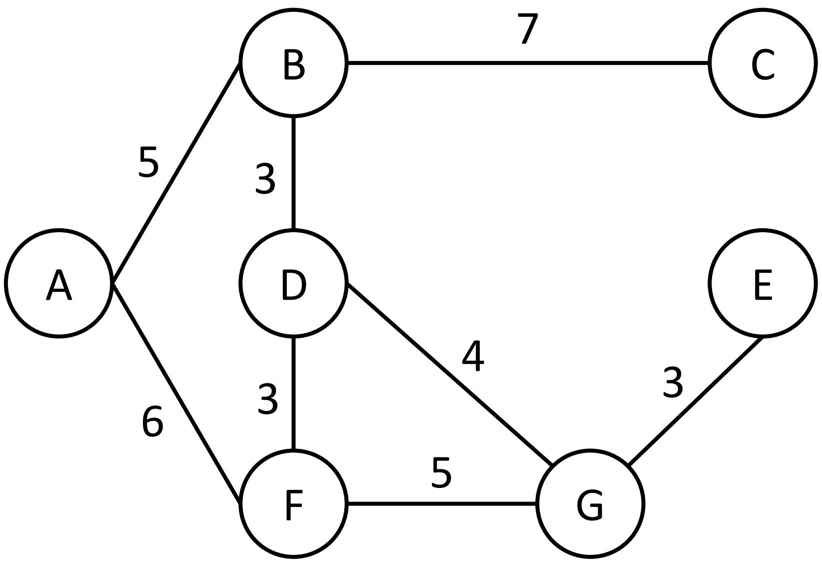

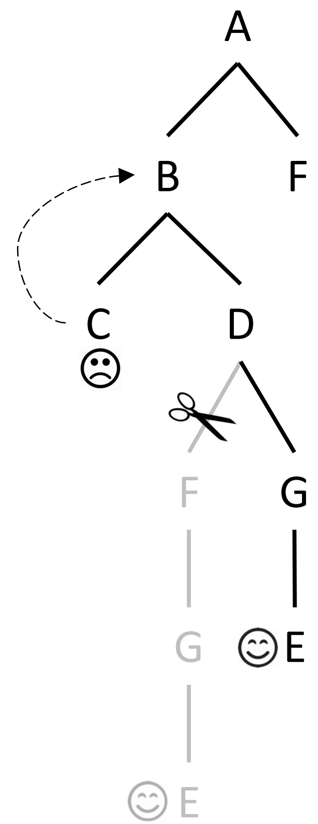

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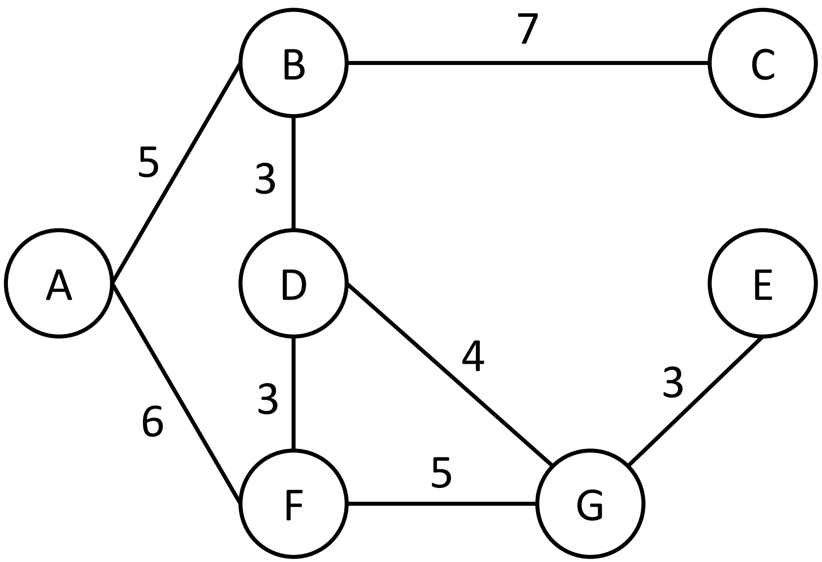
BFS with Enqueued List



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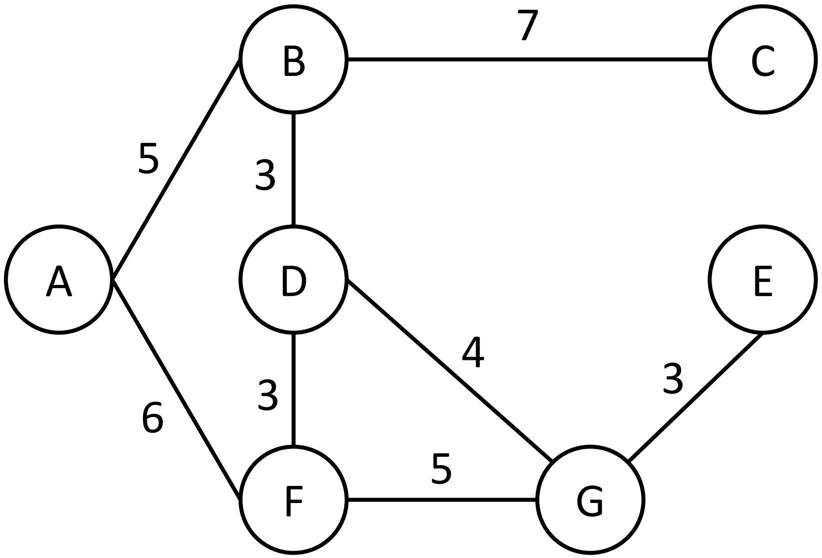

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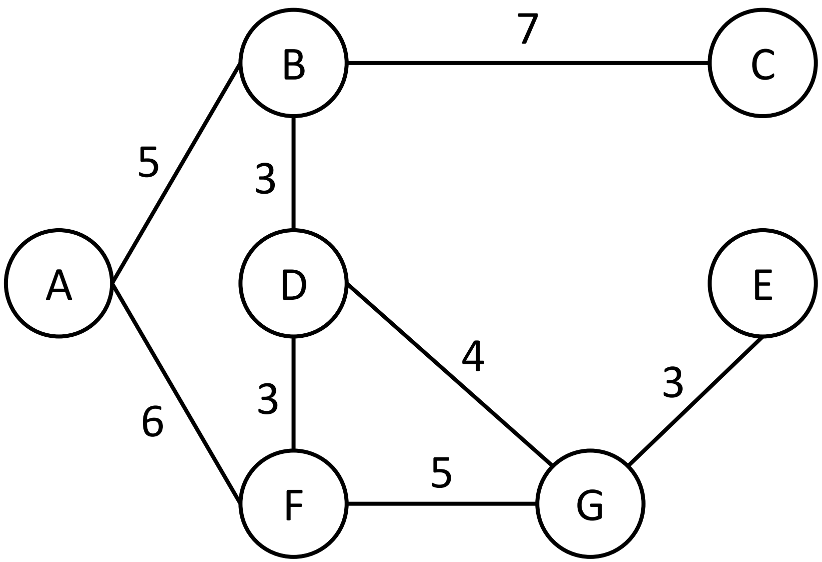

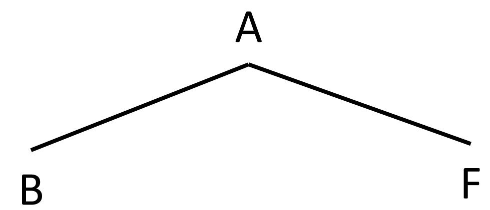

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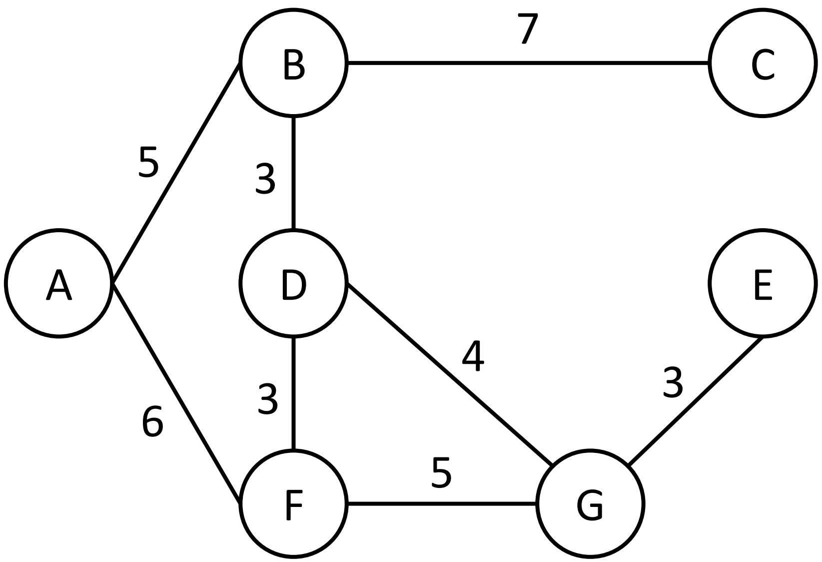

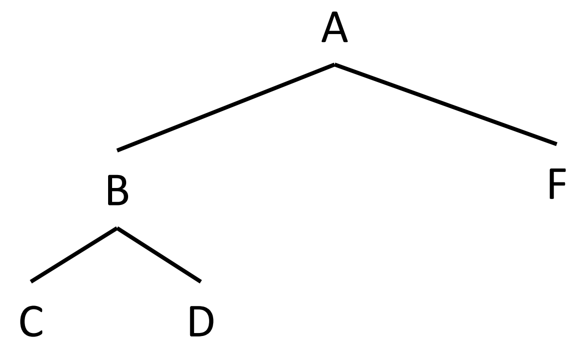

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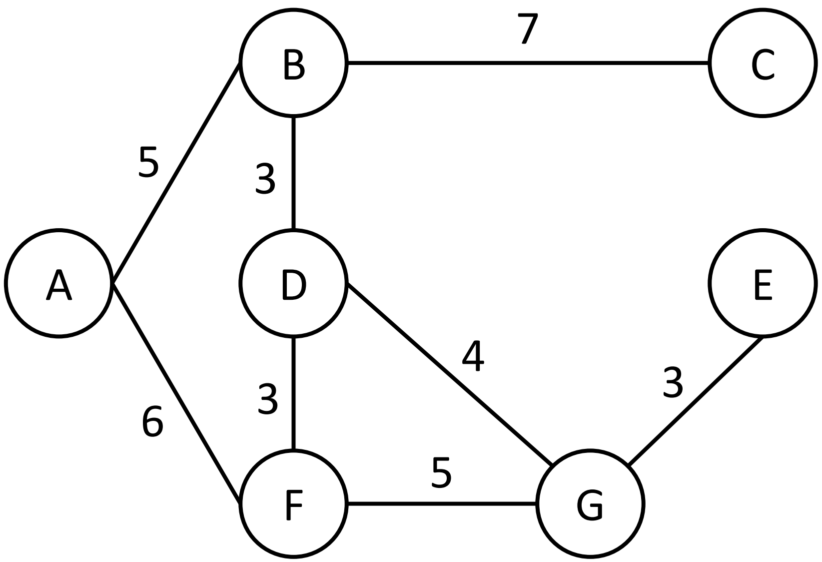

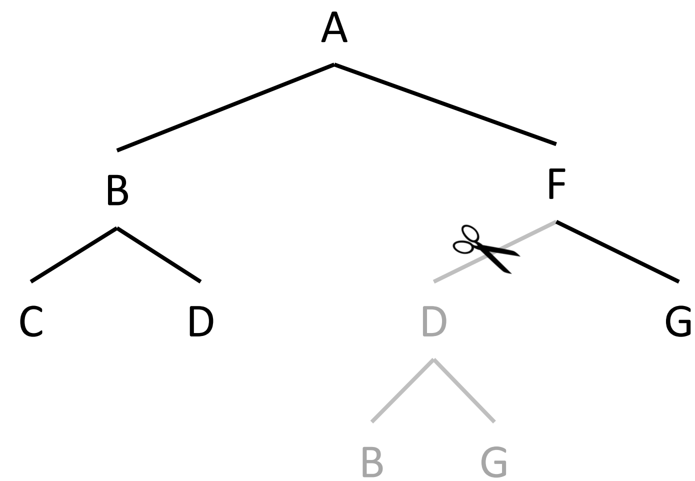

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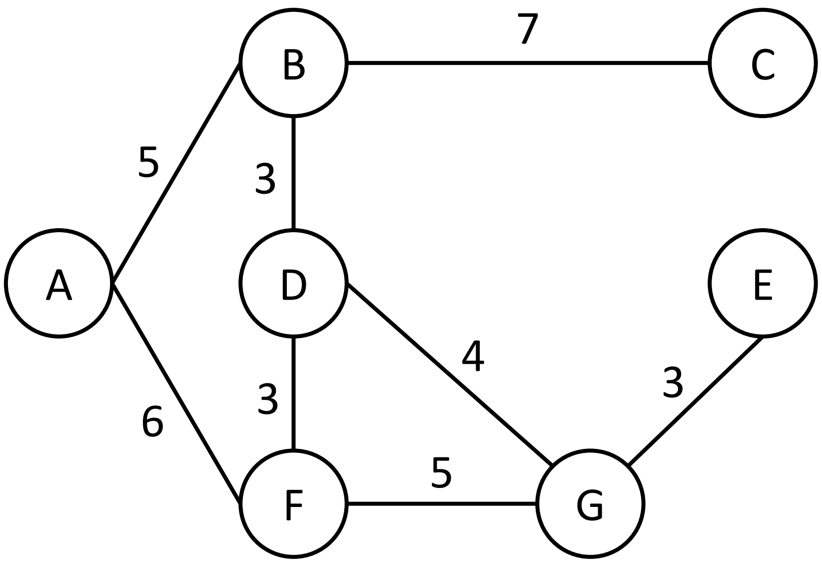

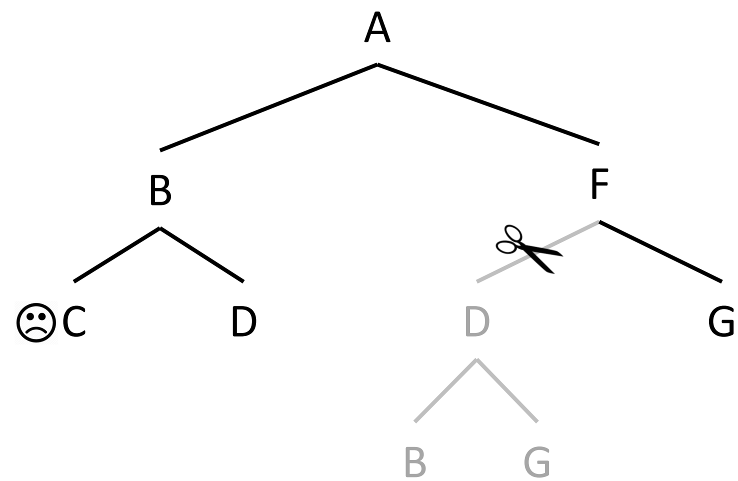

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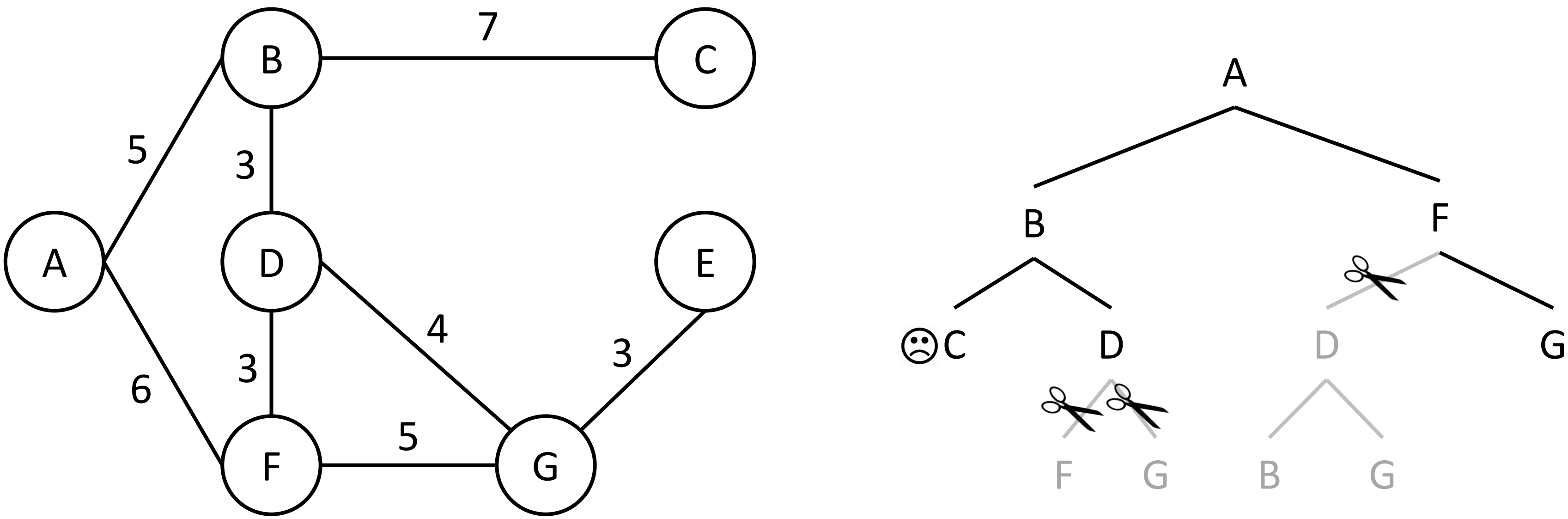

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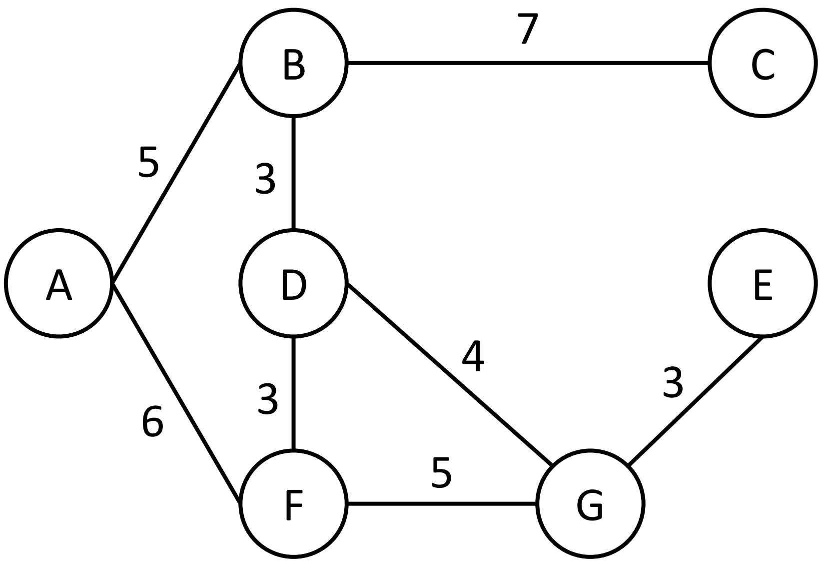
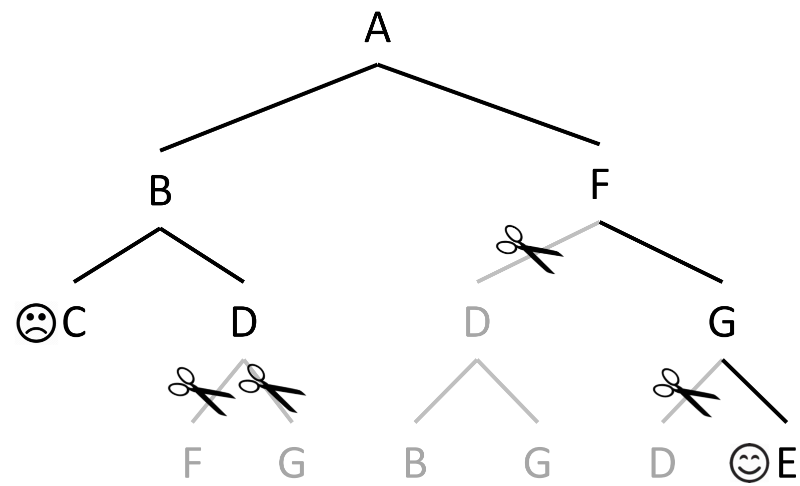
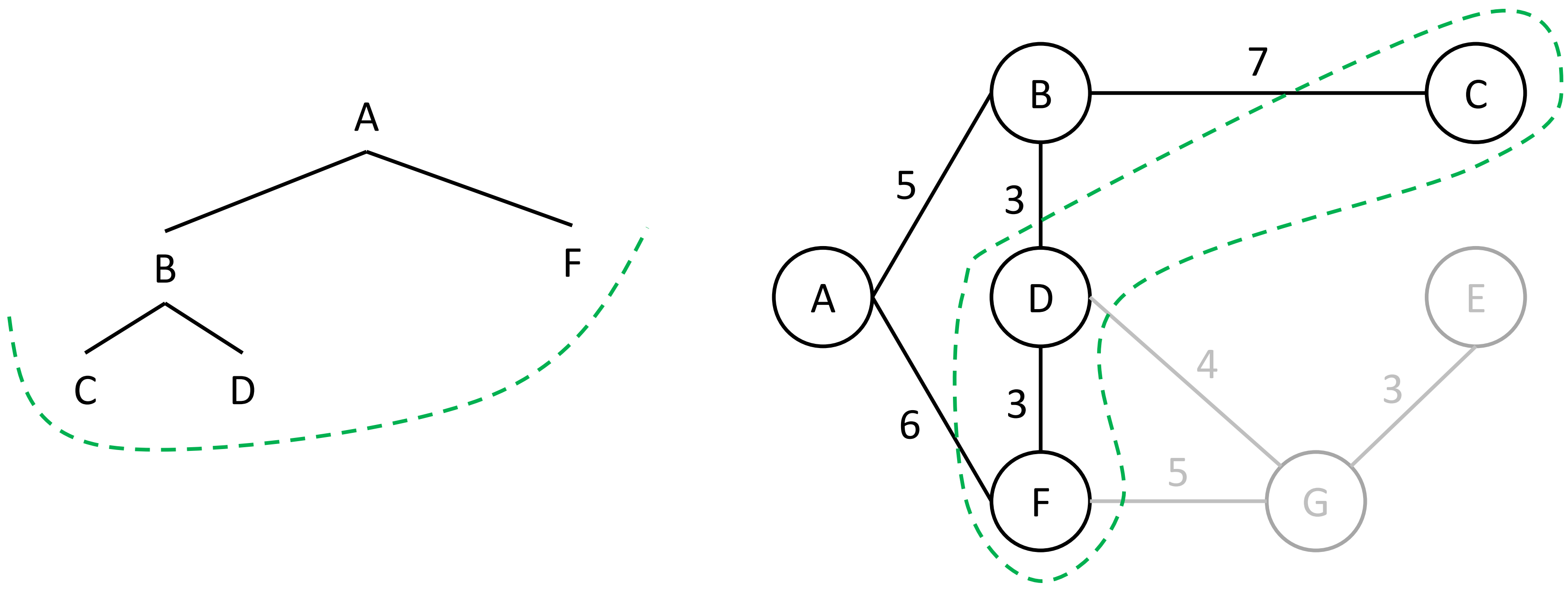




Implementation

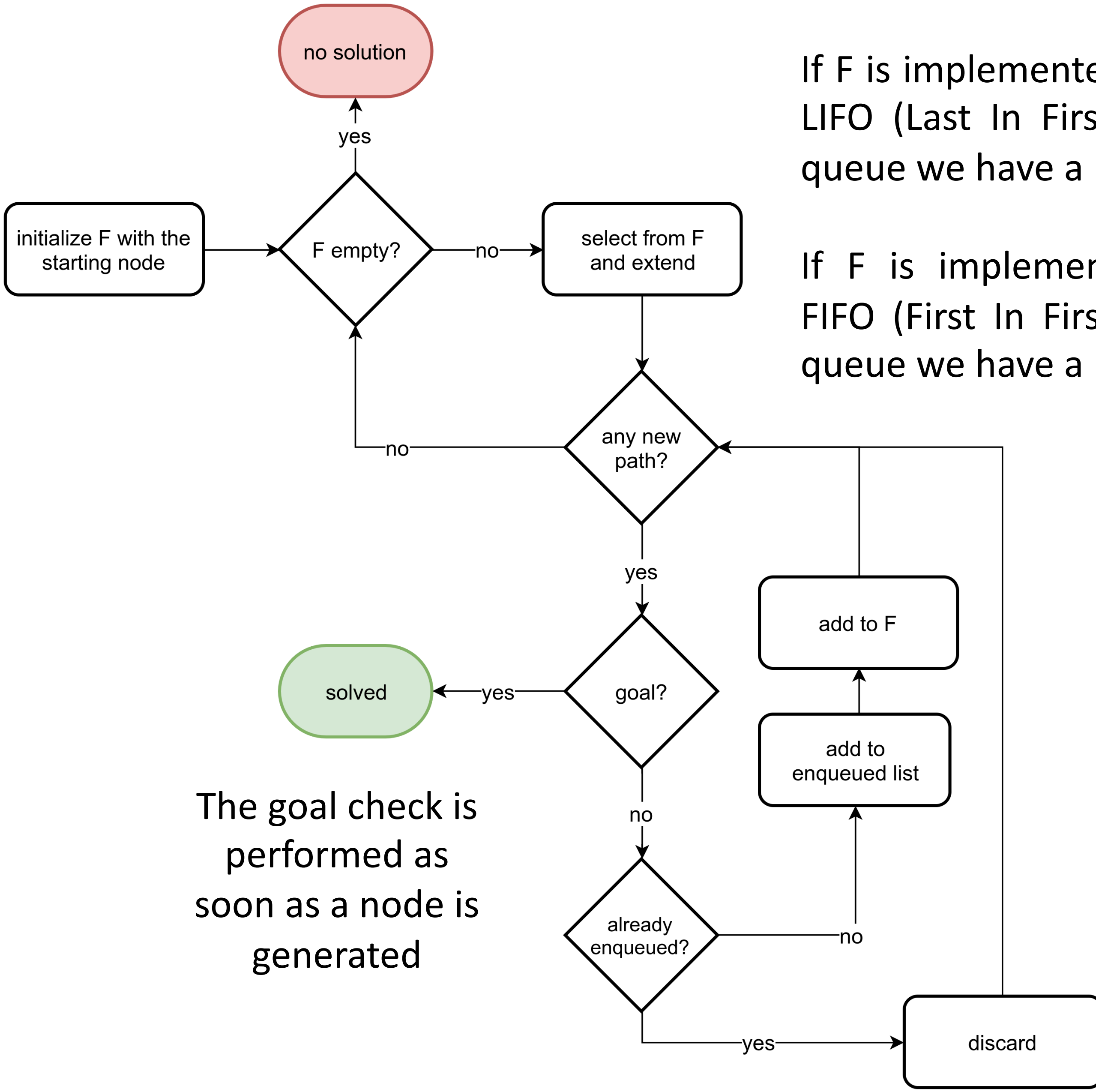
- The implementation of the previous algorithms is based on two data structures:
  - A queue **F** (Frontier), elements ordered by priority, a selection consumes the element with highest priority
  - A list **EL** (Enqueued List, nodes that have already been put on the tree)

- The frontier F contains the terminal nodes of all the paths currently under exploration on the tree



- The frontier **separates** the explored part of the state space from the unexplored part
- In order to reach a new unexplored state, we need to pass from the frontier (separation property)

Implementation



If F is implemented as a LIFO (Last In First Out) queue we have a DFS

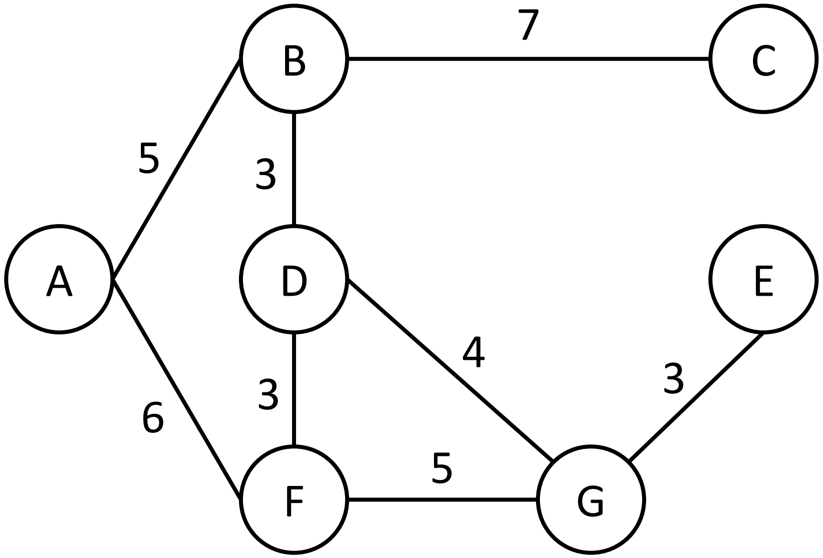
If F is implemented a FIFO (First In First Out) queue we have a BFS

The goal check is performed as soon as a node is generated

Search for the optimal solution

- Now we assume to be interested in the solution with minimum cost (not just any path to the goal, but the cheapest possible)
- To devise an optimal search algorithm we take the moves from BFS. Why it seems reasonable to do that?
- We generalize the idea of BFS to that of Uniform Cost Search (UCS)
- BFS proceeds by *depth* levels, UCS does that by *cost* levels (as a consequence, if costs are all equal to some constant BFS and UCS coincide)
- Cost accumulated on a path from the start node to v: $g(v)$ (we should include a dependency on the path, but it will always be clear from the context)
- For now let’s remove the enqueued list and the goal checking as we know it

Uniform Cost Search (UCS)



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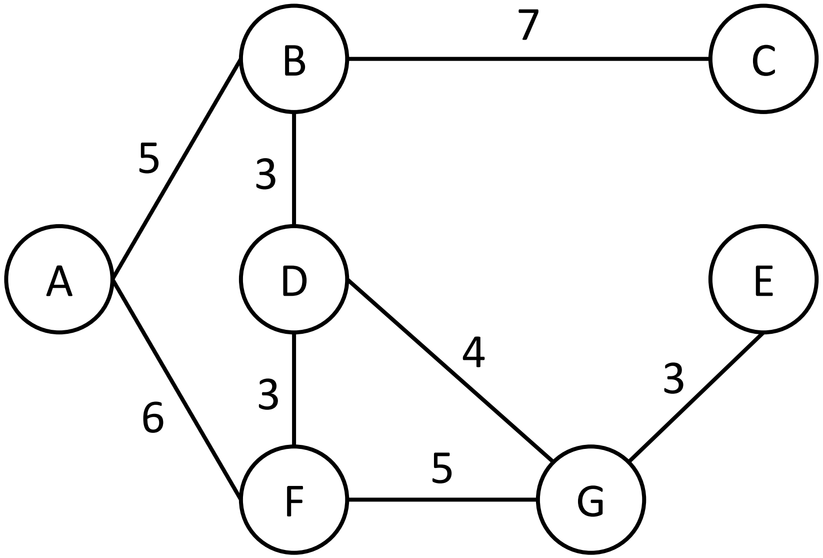

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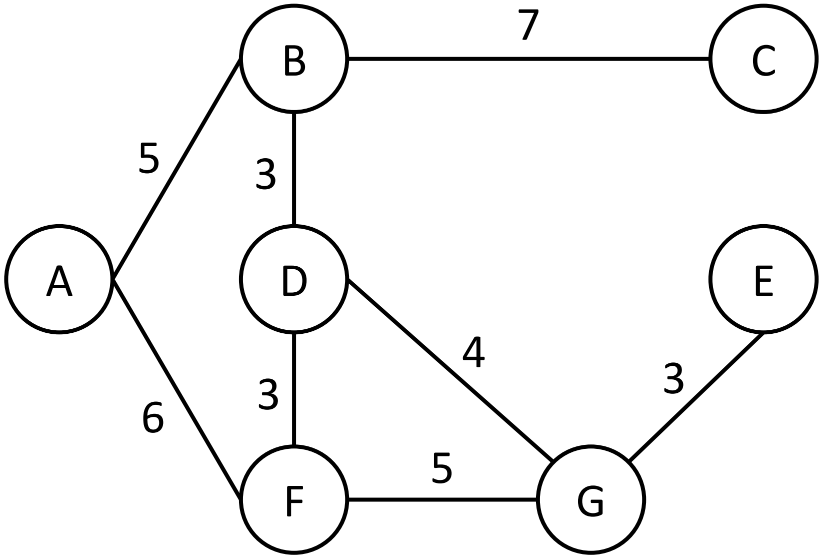
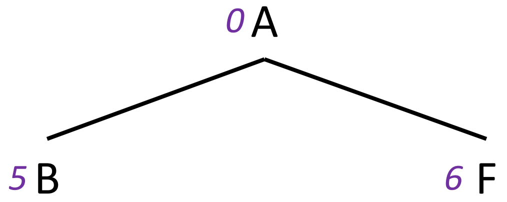
Uniform Cost Search (UCS)





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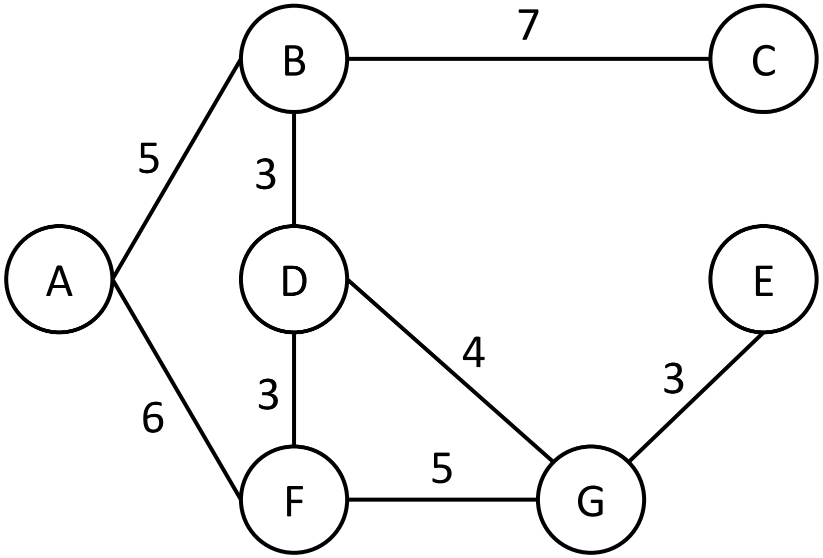

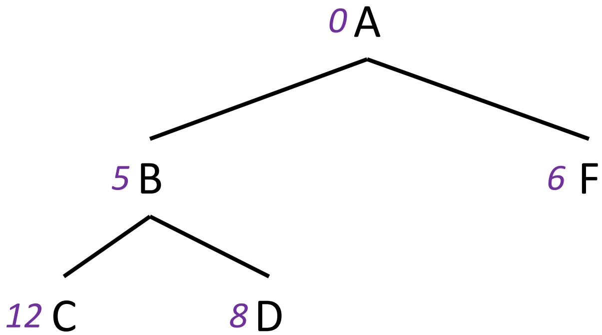

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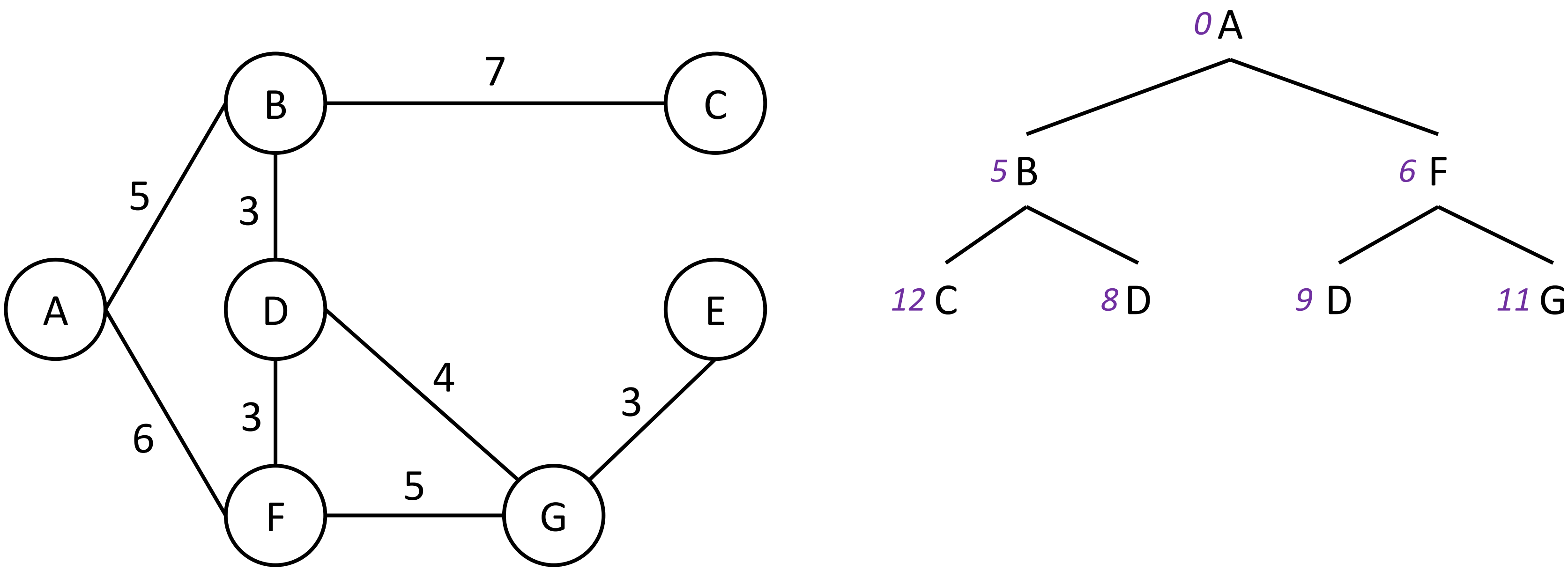

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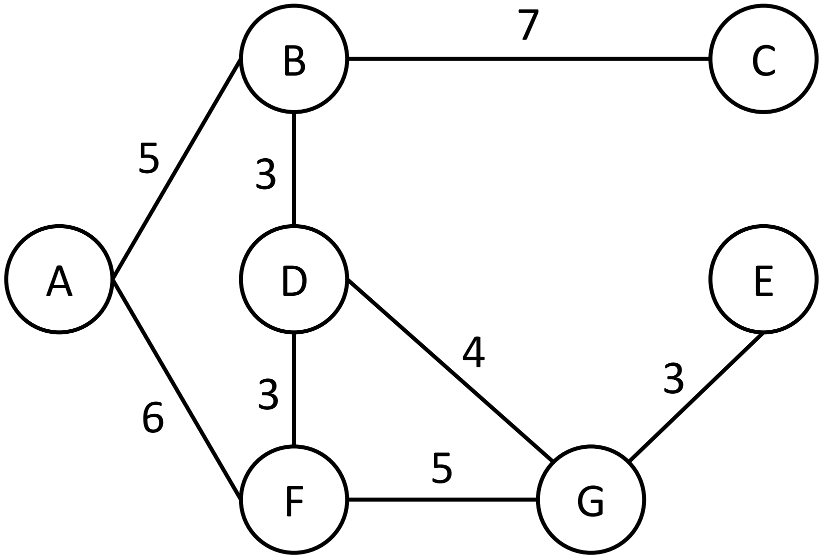

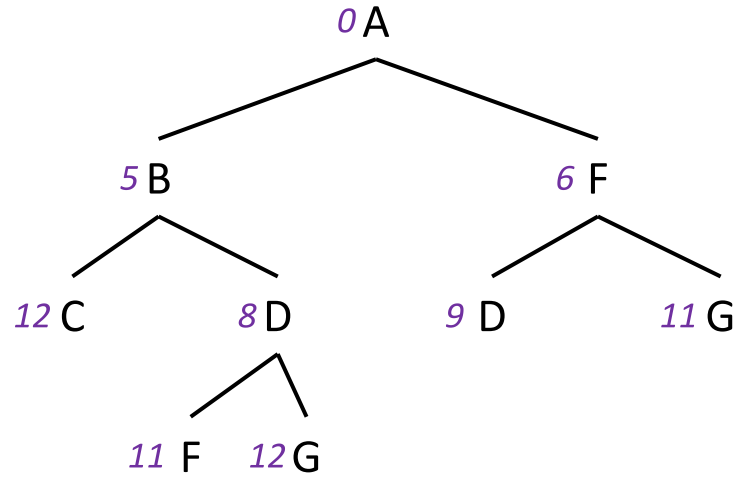

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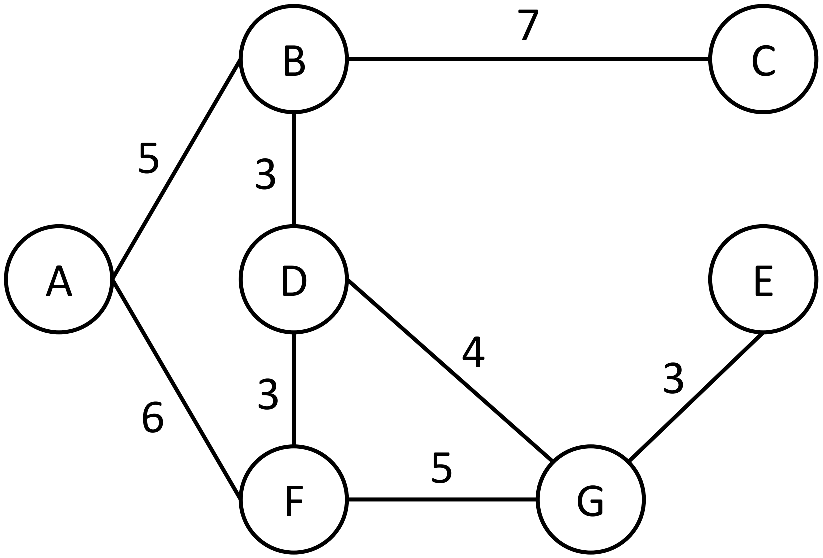

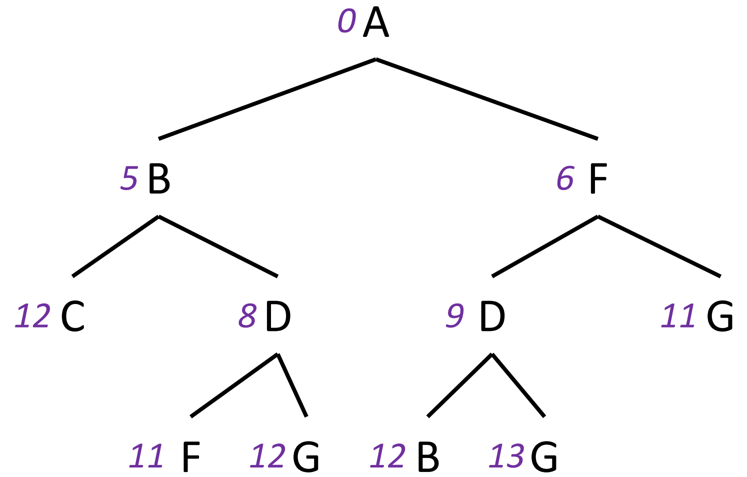

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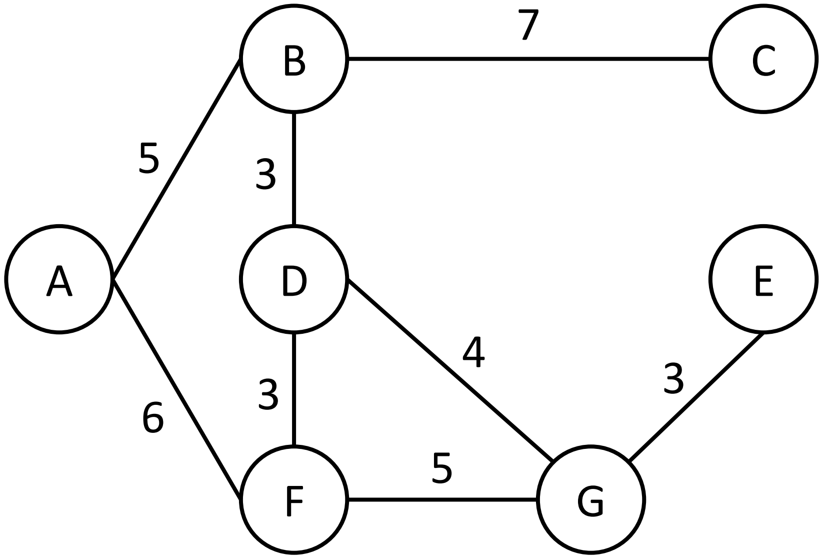

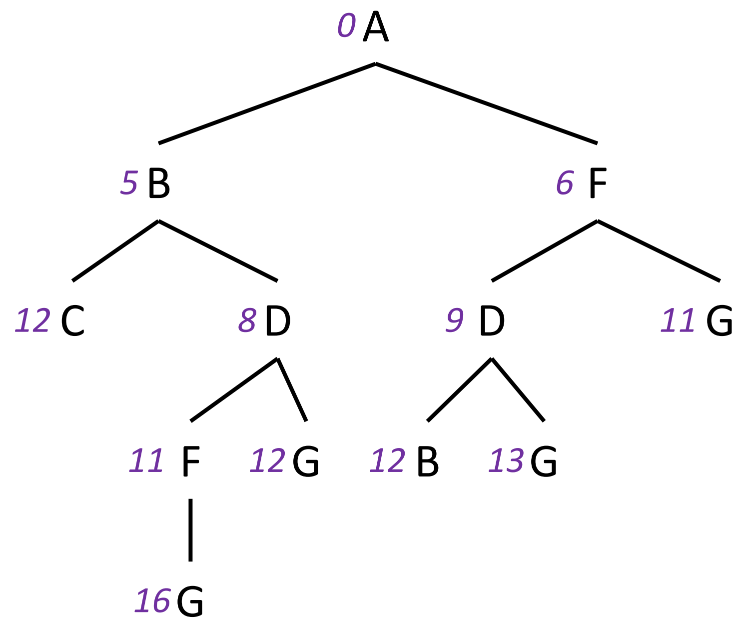

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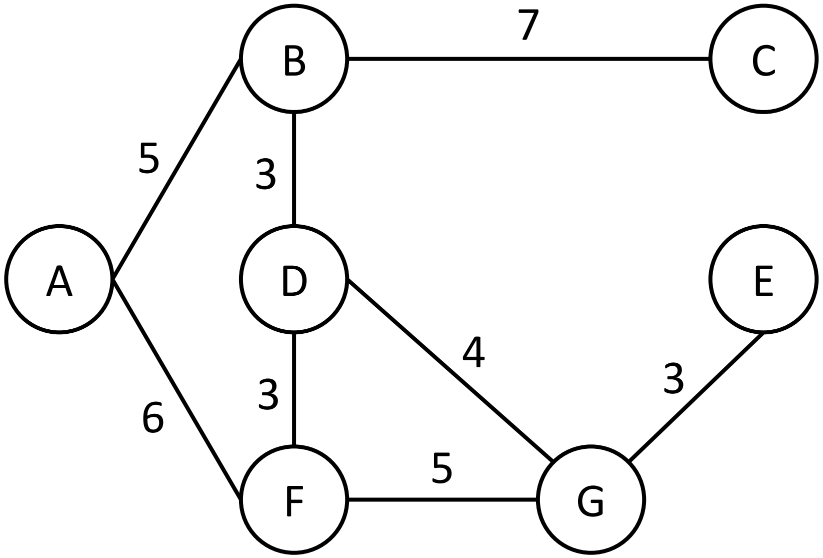

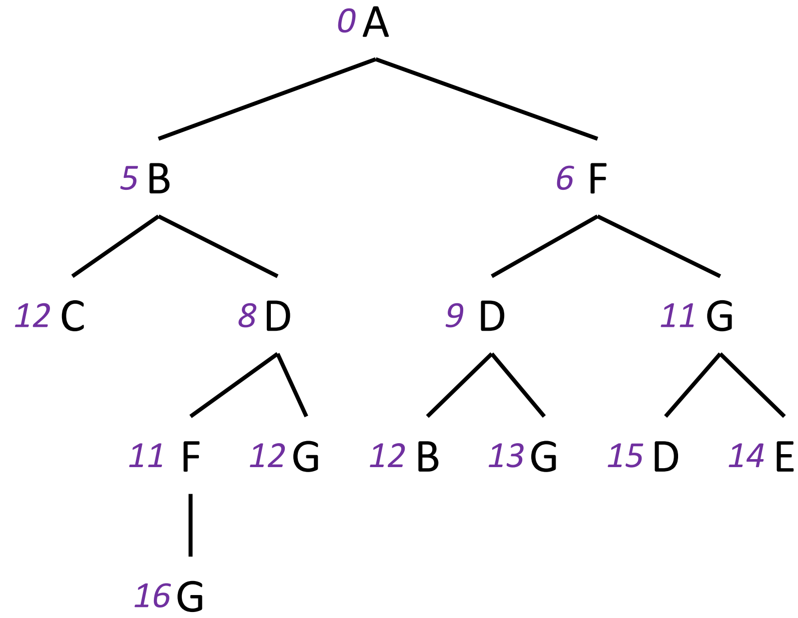

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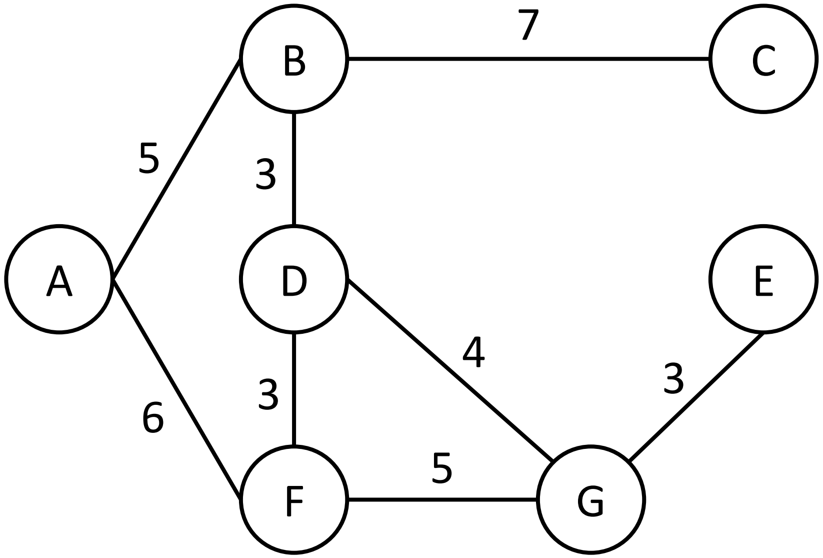

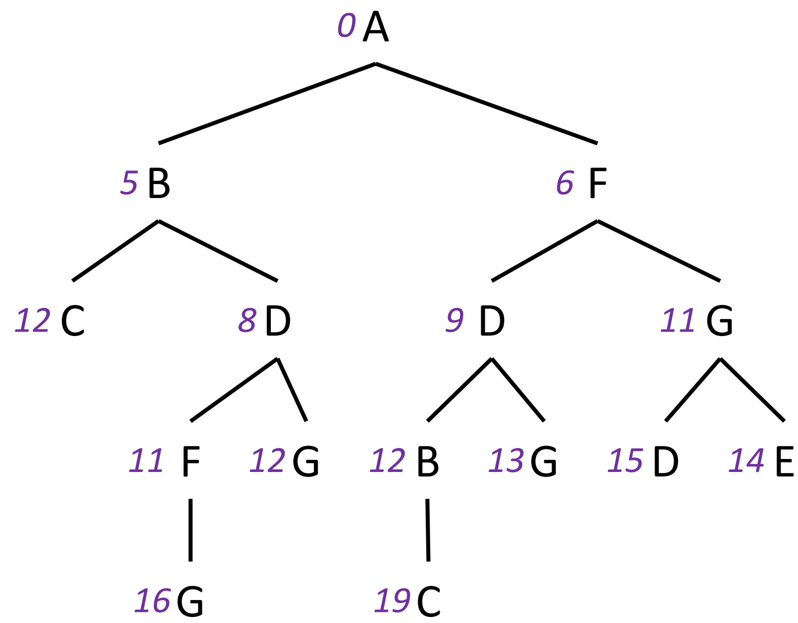

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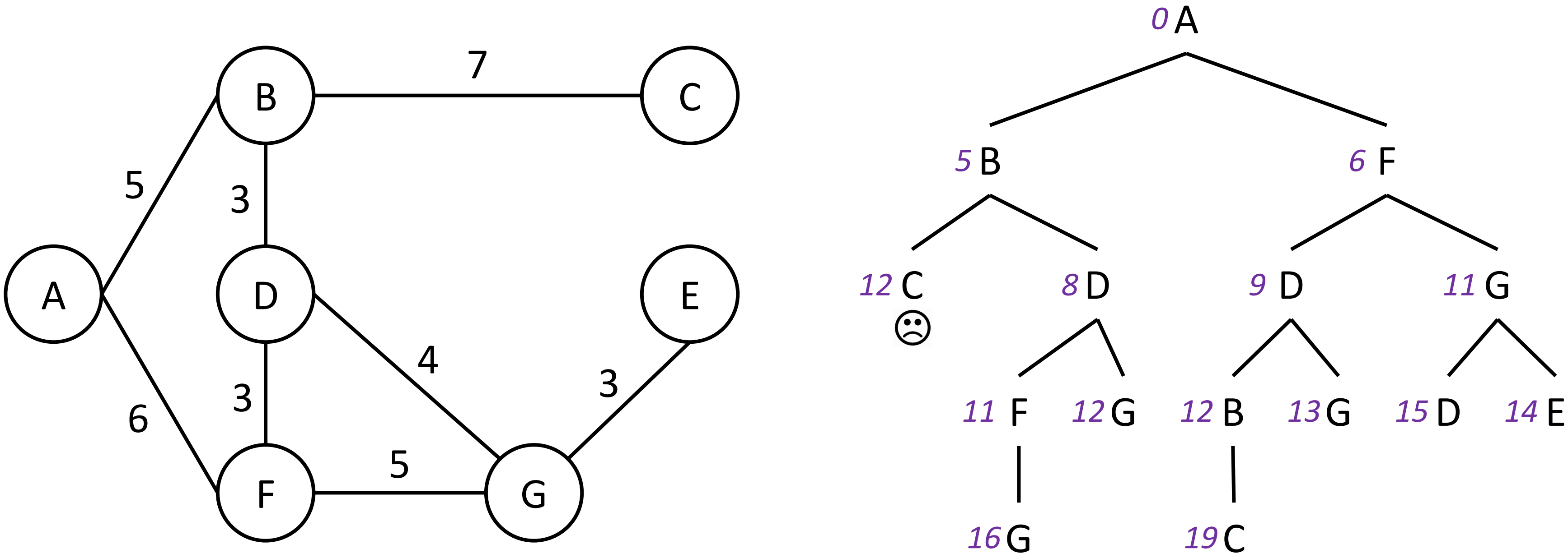

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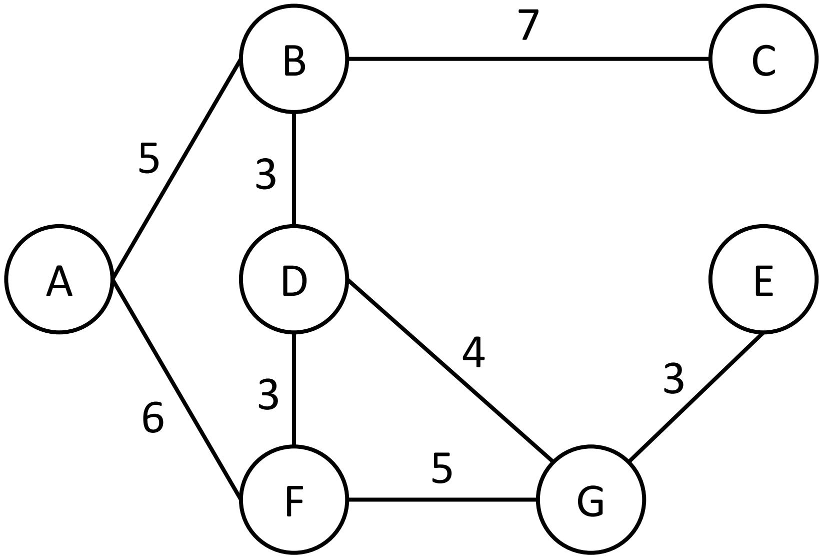

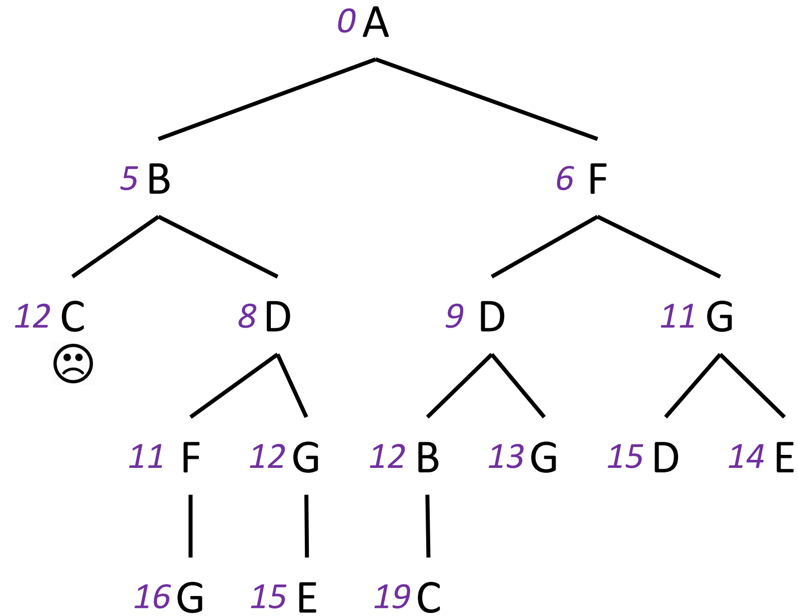

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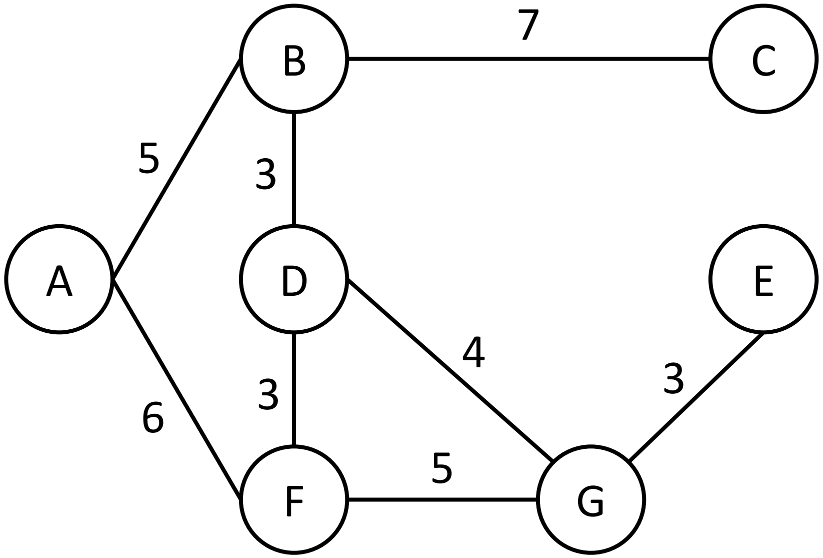

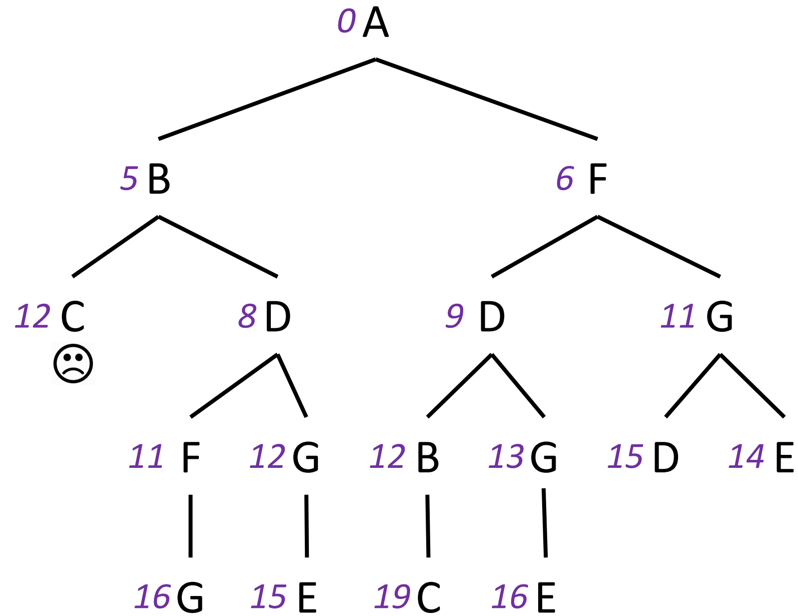

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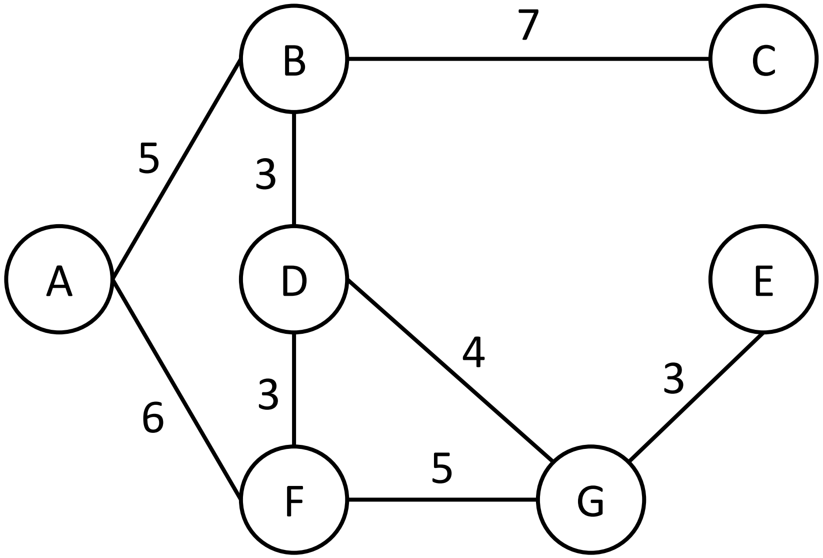

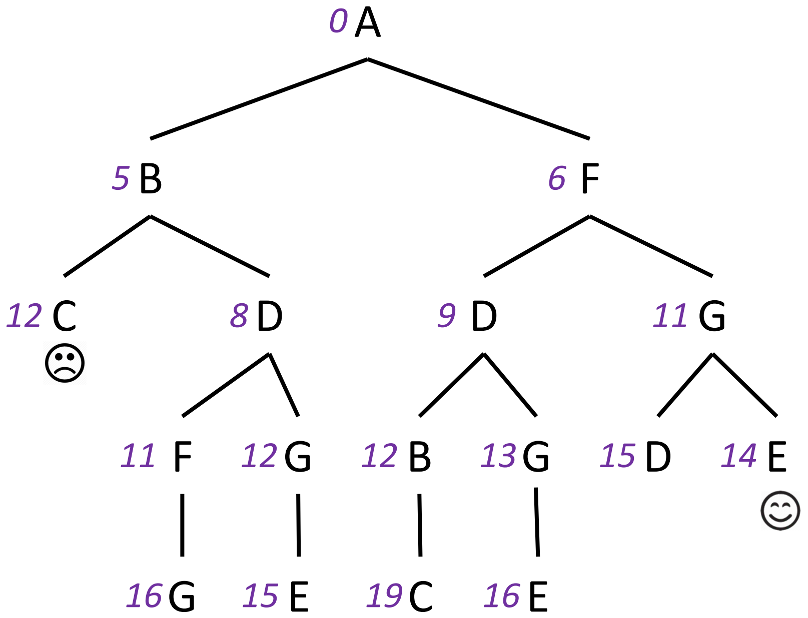

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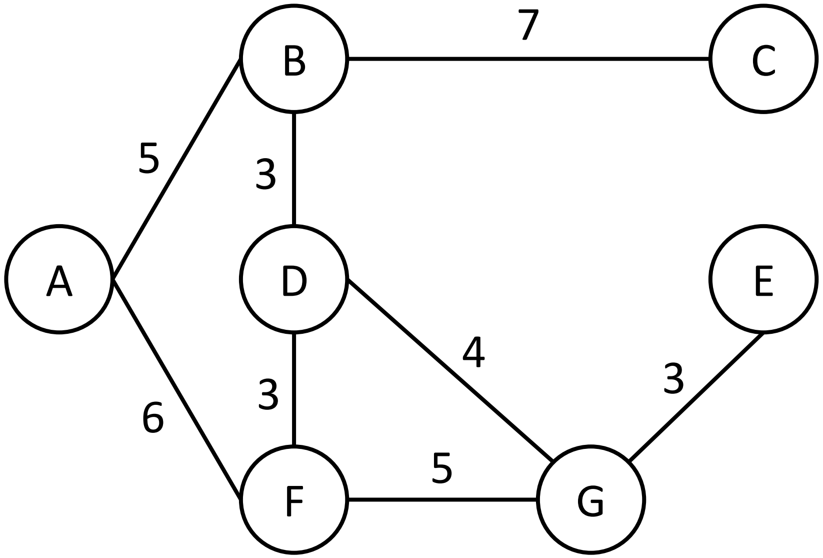
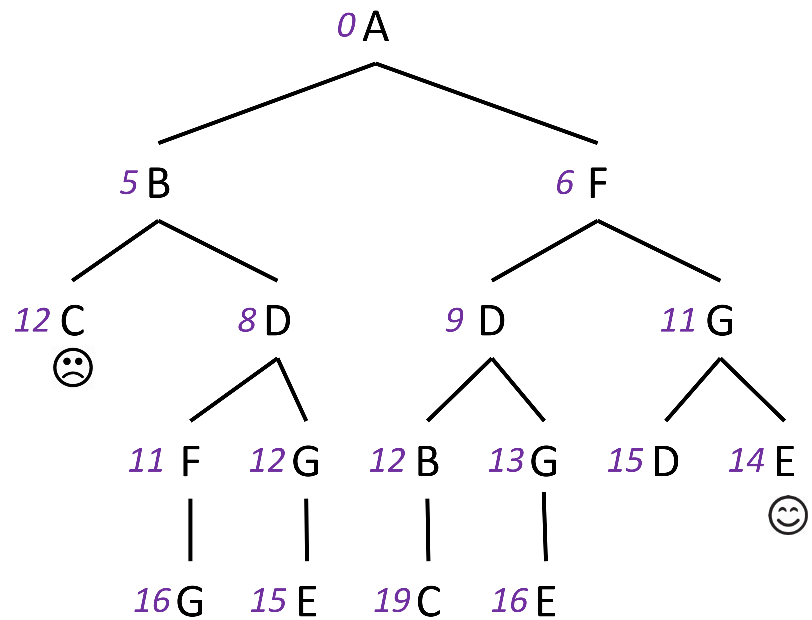


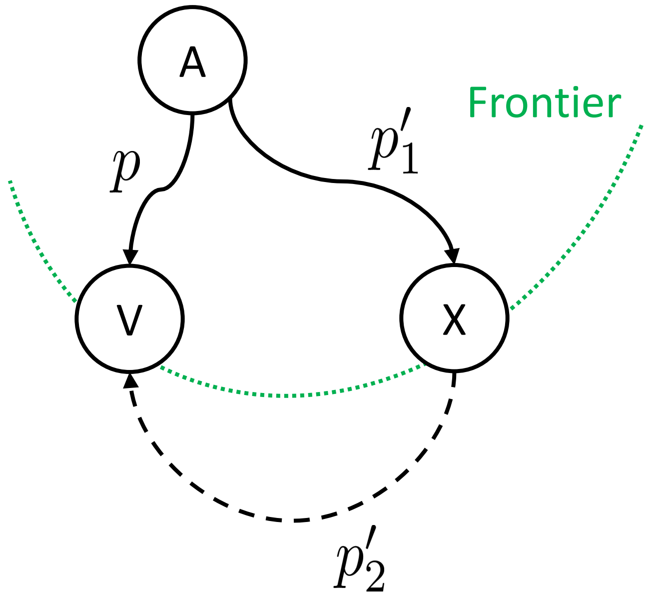


- Have we found the optimal path to the goal? In this problem instance, we can answer *yes* by inspecting the graph
- How about larger instances? Can we prove optimality?
- Actually, we can prove a stronger claim: every time UCS selects **for the first time** a node for expansion, the associated path leading to that node has minimum cost

Optimality of UCS

Hypotheses:

1. UCS selects from the frontier a node V that has been generated through a path p
2. p is not the optimal path to V



Given 2 and the frontier separation property, we know that there must exist a node X on the frontier, generated through a path p'_1 that is on the optimal path $p' \neq p$ to V; let assume $p' = p'_1 + p'_2$

$c(p') = c(p'_1) + c(p'_2) < c(p)$ since, from Hp, p' is optimal

$c(p'_1) < c(p'_1) + c(p'_2) < c(p)$ since costs are positive

$c(p'_1) < c(p)$ X would have been chosen before V, then 1 is false

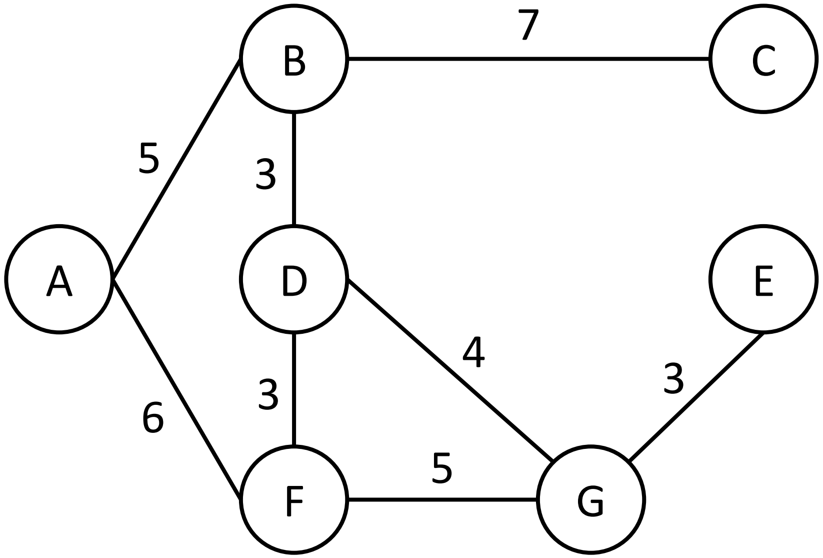
Optimality of UCS

If when we select for the first time we discover the optimal path, there is no reason to select the same node a second time: **extended list**

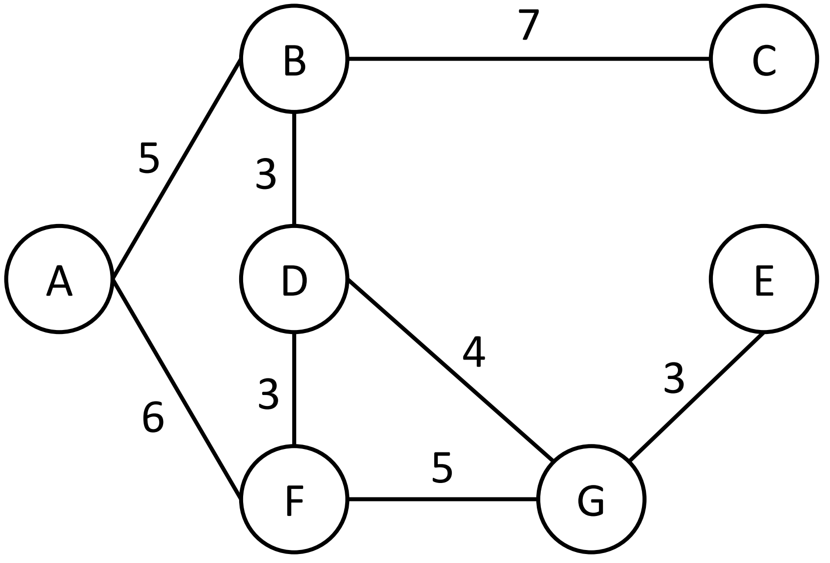
Every time we select a node for extension:
- If the node is already in the extended list we discard it
- Otherwise we extend it and we put it the extended list

- (Warning: we are not using an enqueued list, it would actually make the search not sound!)

UCS with extended list

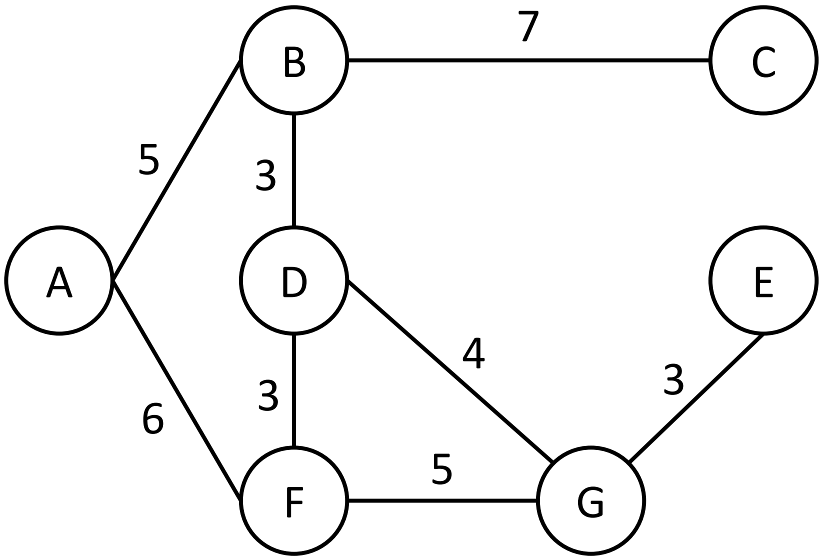


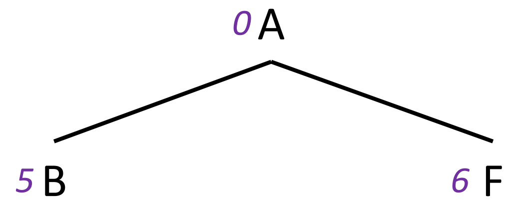
UCS with extended list



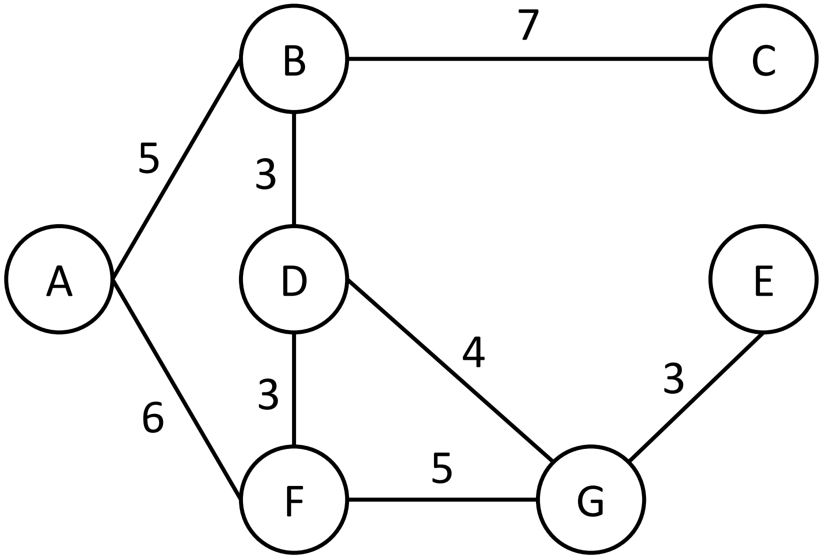
0 A

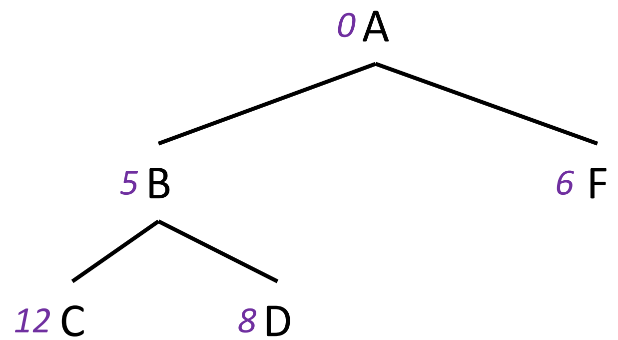
UCS with extended list



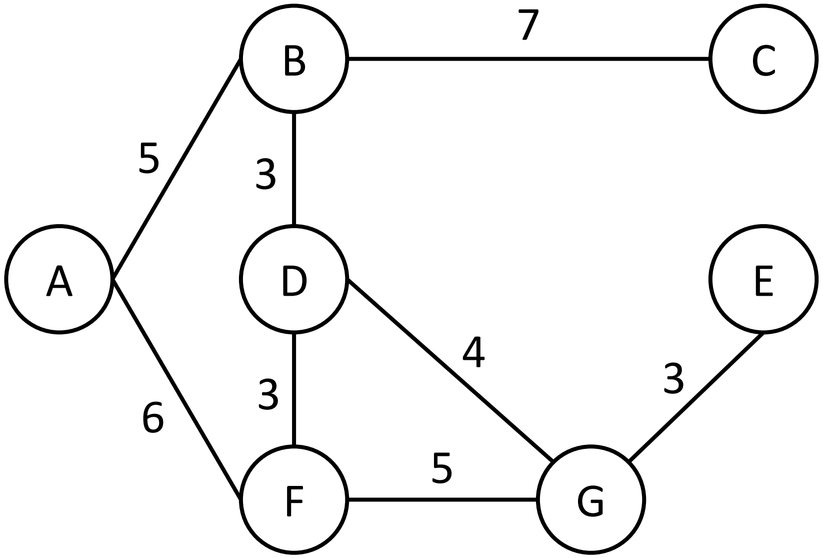


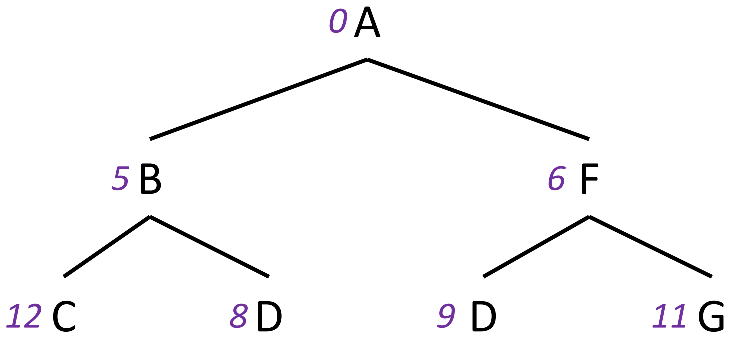
UCS with extended list



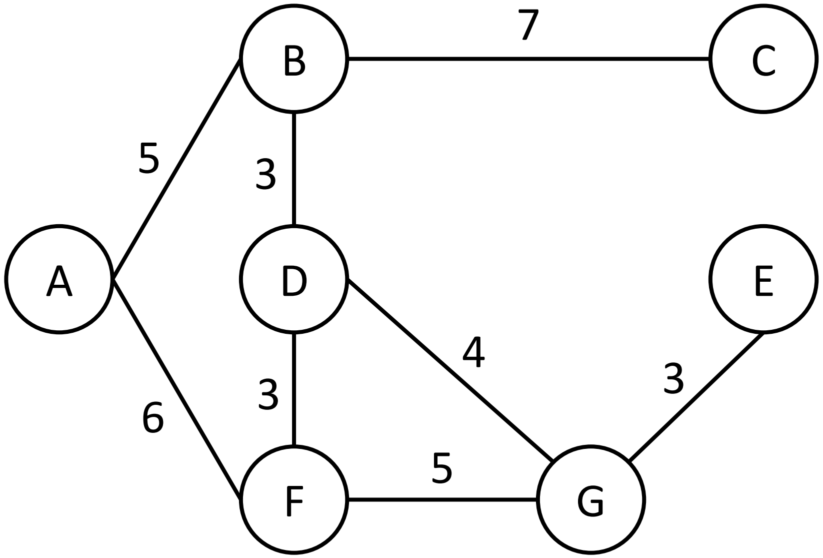


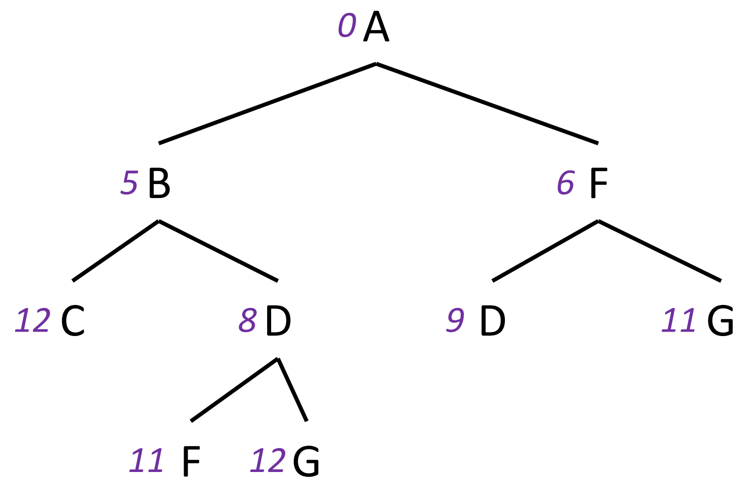
UCS with extended list



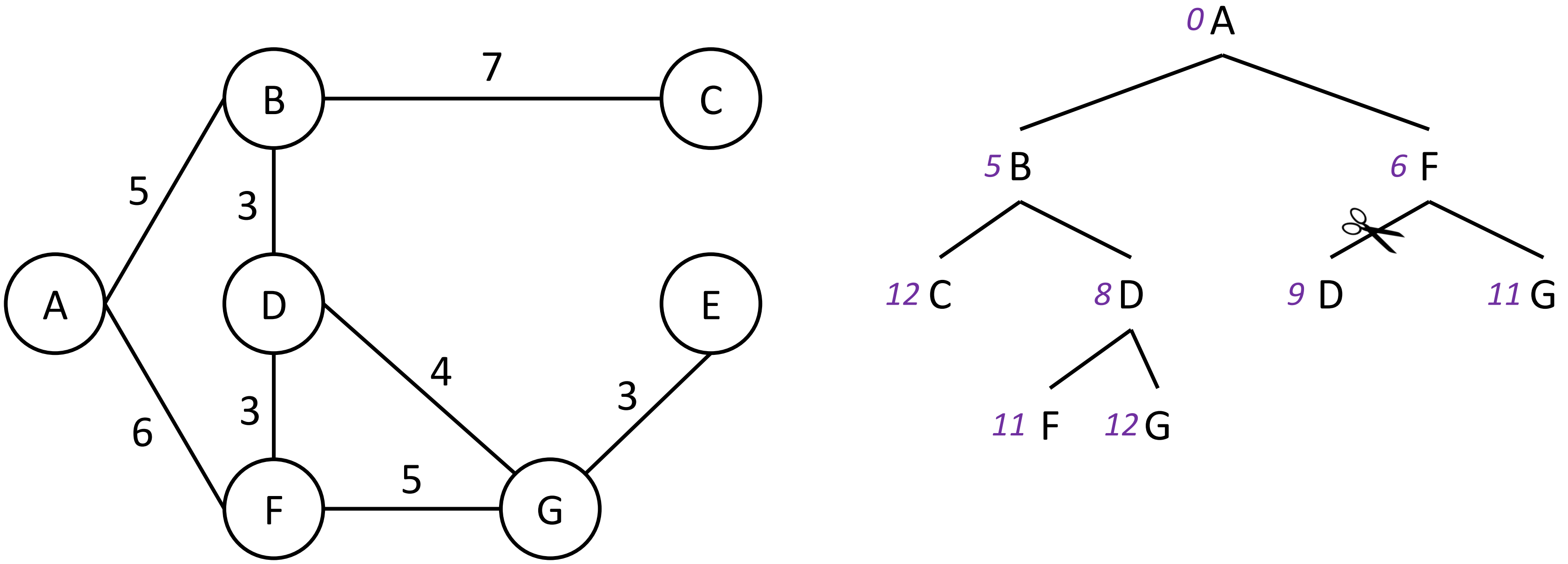


UCS with extended list





UCS with extended list



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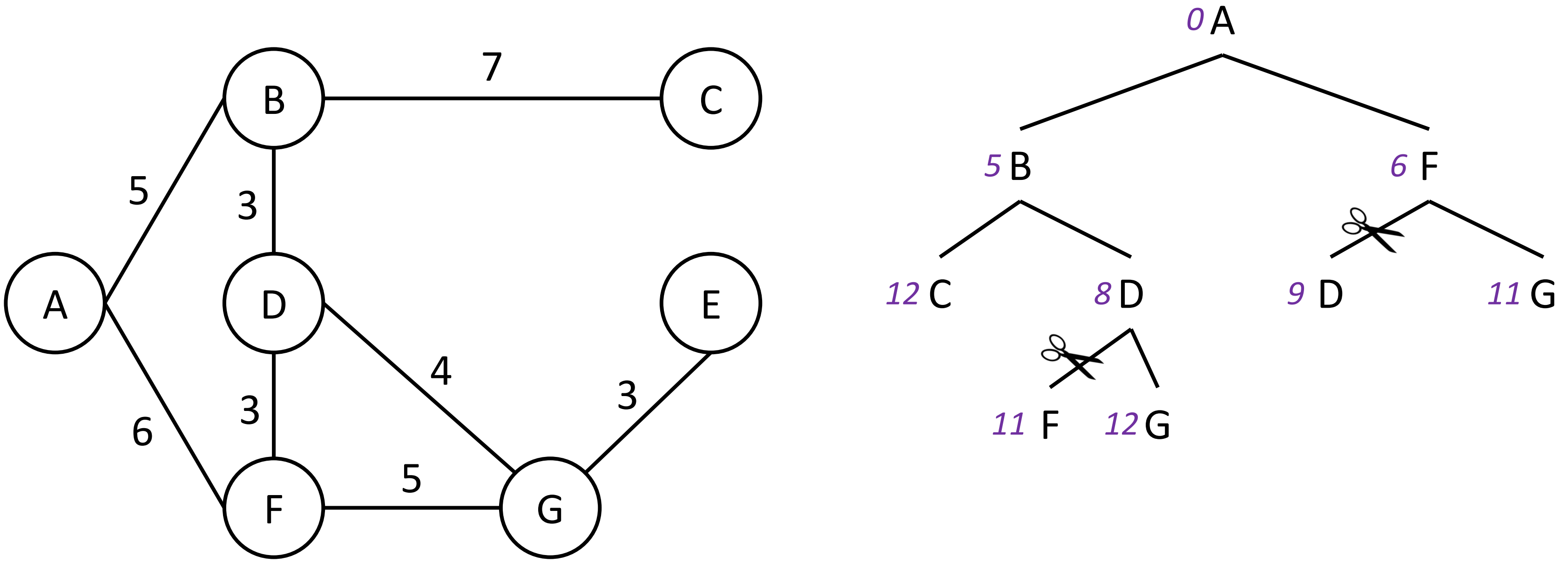

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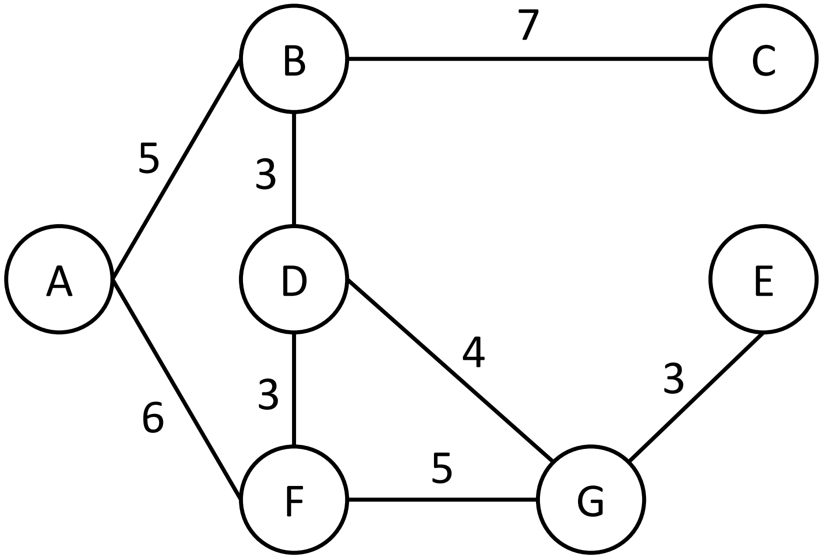
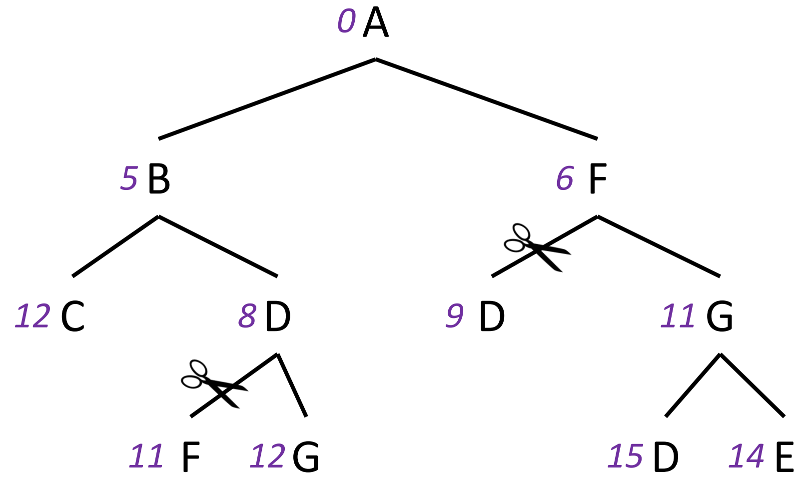
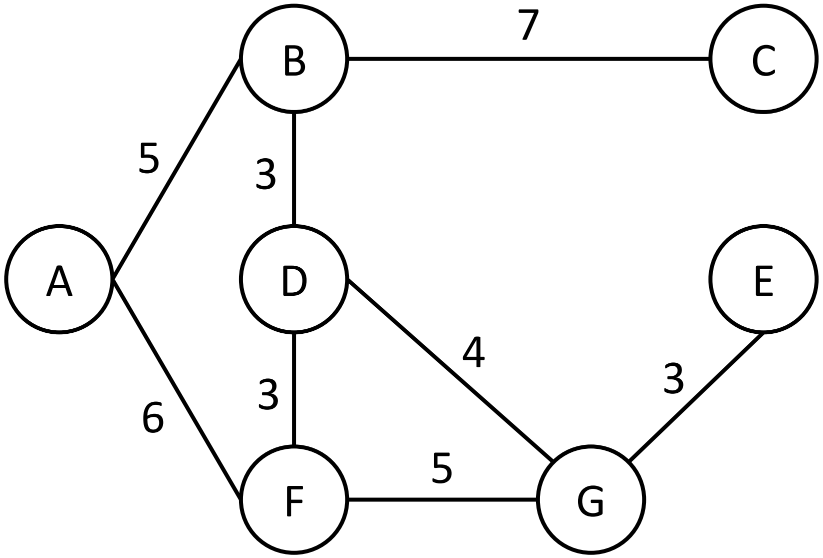
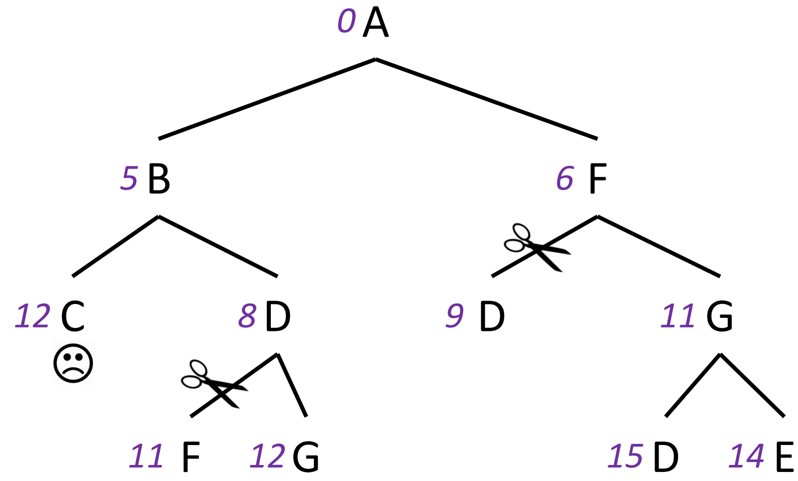


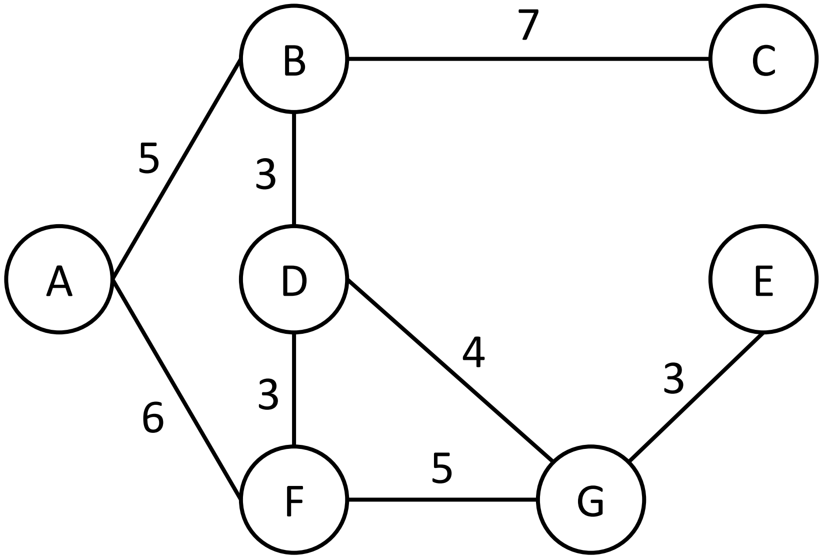


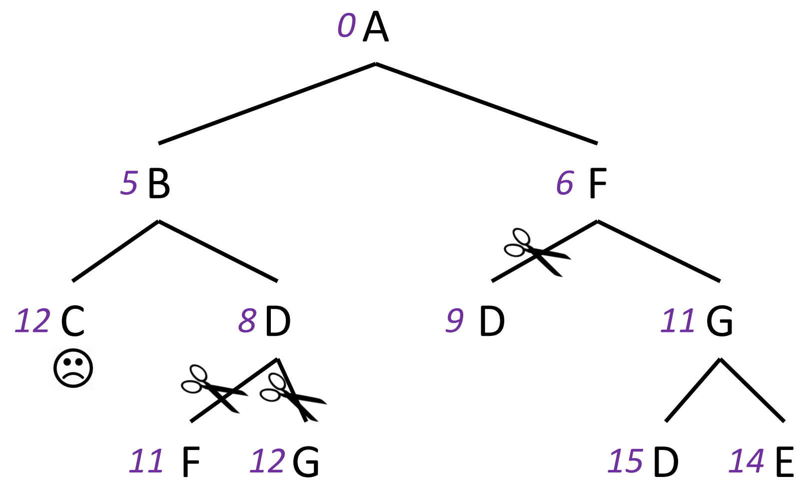
UCS with extended list





UCS with extended list





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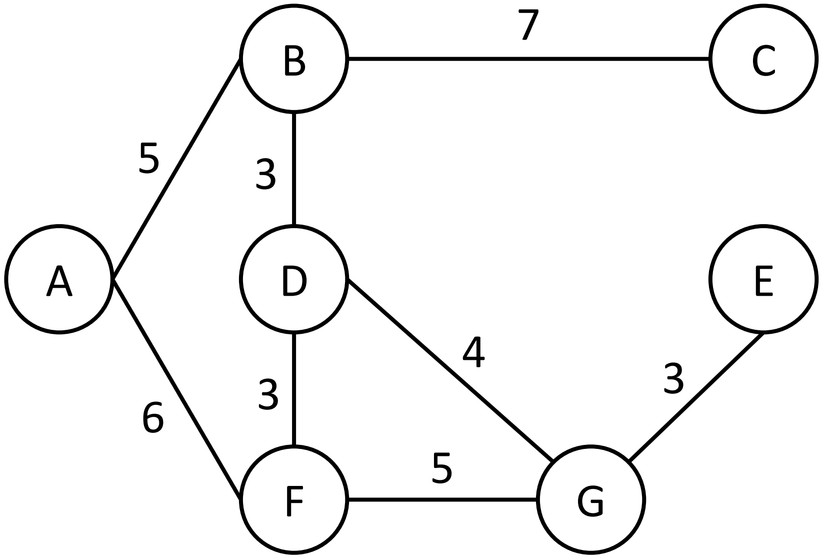
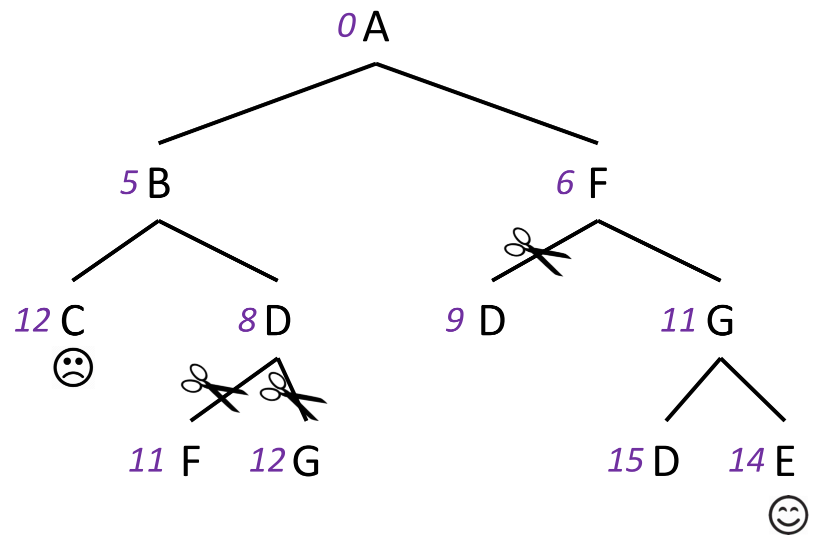
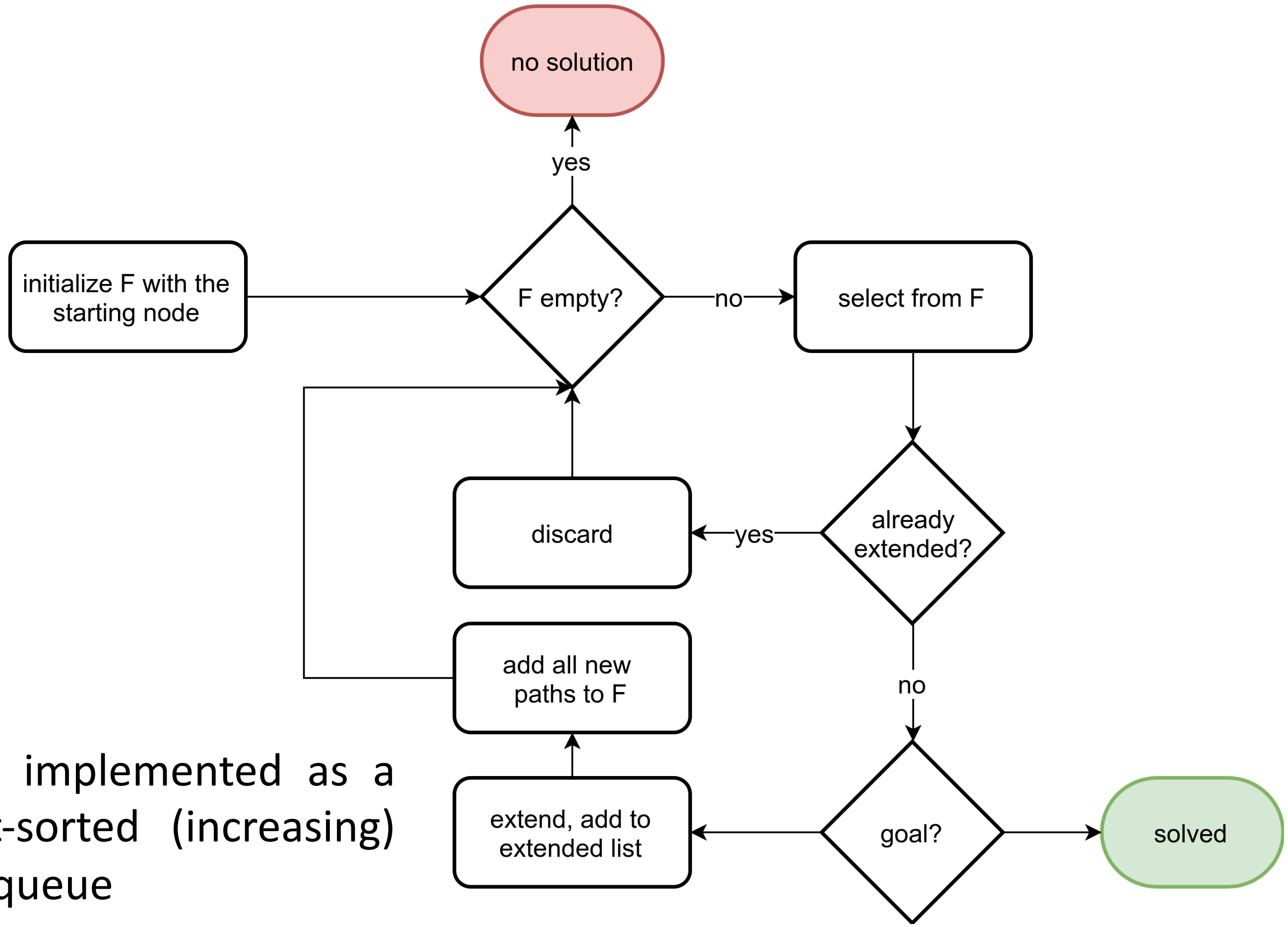




- Thanks to the extended list we can prune two branches

Implementation

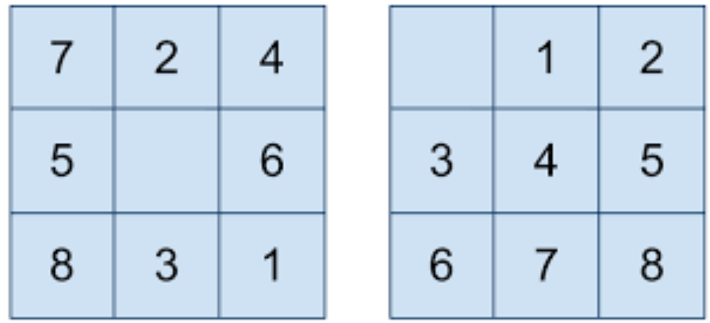


F is implemented as a cost-sorted (increasing) list queue

The goal check is done when the node is selected (not when is generated)

- Question: is this search informed?

Discrete Search Problems: 8-Puzzle



Start State   Goal State

- States: location of each digits in the eight tiles + blank one
- Initial State
- Goal State
- Actions: Left, Right, Up, Down
- Transition: given a state and an action, the resulting board
- Goal Test: if the states are equal to the goal state
- Cost: each movement costs 1, the lowest number of tile move the lowest the cost

Discrete Search Problems: 8-Puzzle

7	2	4
5		6
8	3	1

Start State

	1	2
3	4	5
6	7	8

Goal State

- Question: are all states equal?

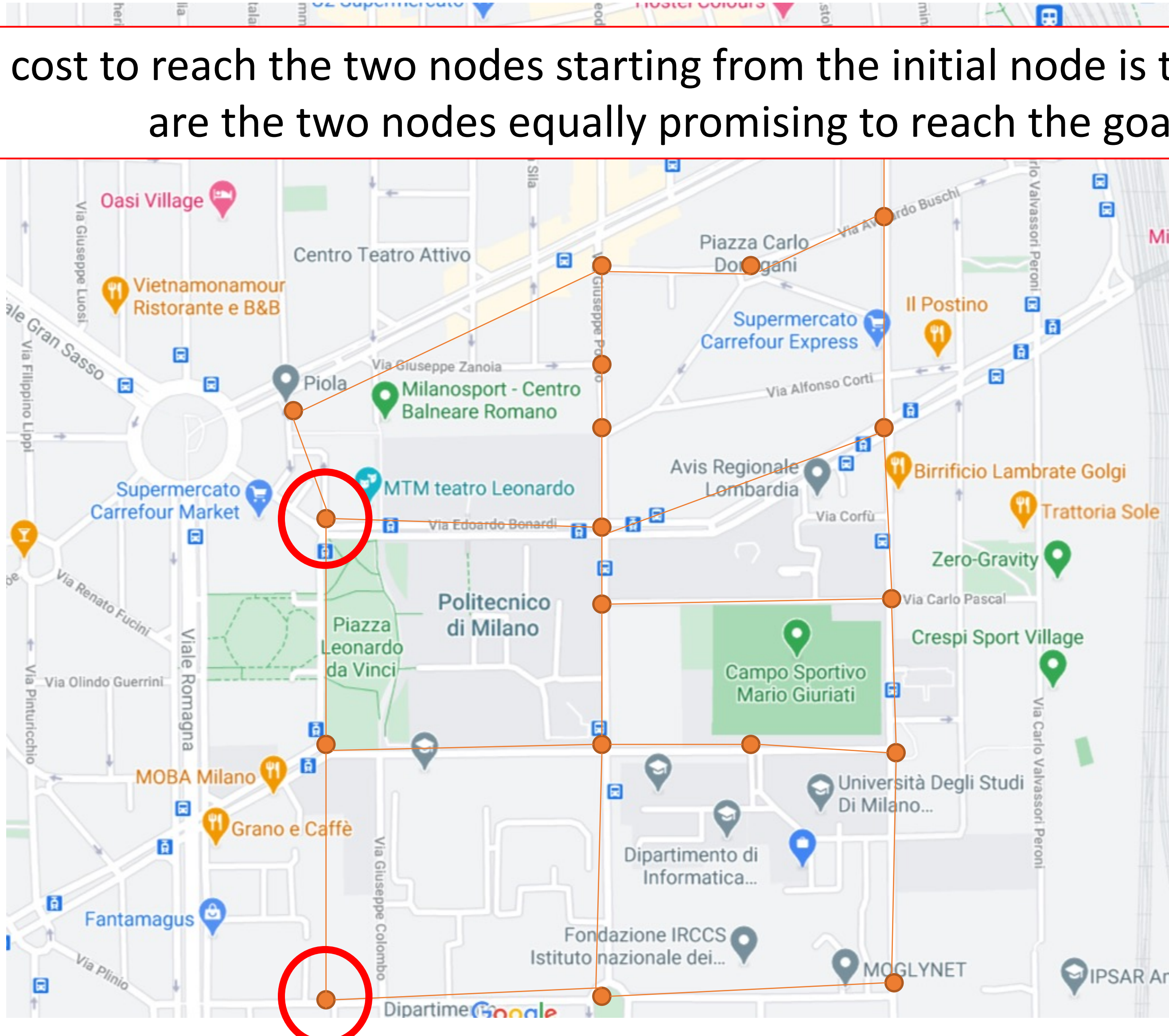
7	2	4
5		6
8	3	1

1	2	3
4		5
8	7	6

1	2	3
4	5	6
8	7	

Example: going home from the CS department with METRO

The cost to reach the two nodes starting from the initial node is the same; but are the two nodes equally promising to reach the goal?



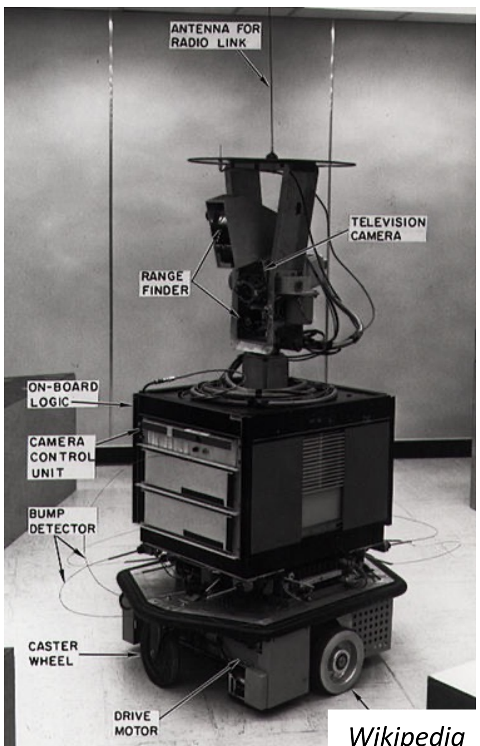
Informed vs non-informed search

- Besides its own rules, any search algorithm decides where to search next by leveraging some knowledge
- **Non-informed** search uses only knowledge specified at problem-definition time (e.g., goal and start nodes, edge costs), just like we saw in the previous examples
- An **informed** search might go beyond such knowledge
- Idea: using an estimate of how far a given node is from the goal
- Such an estimate is often called a **heuristic**

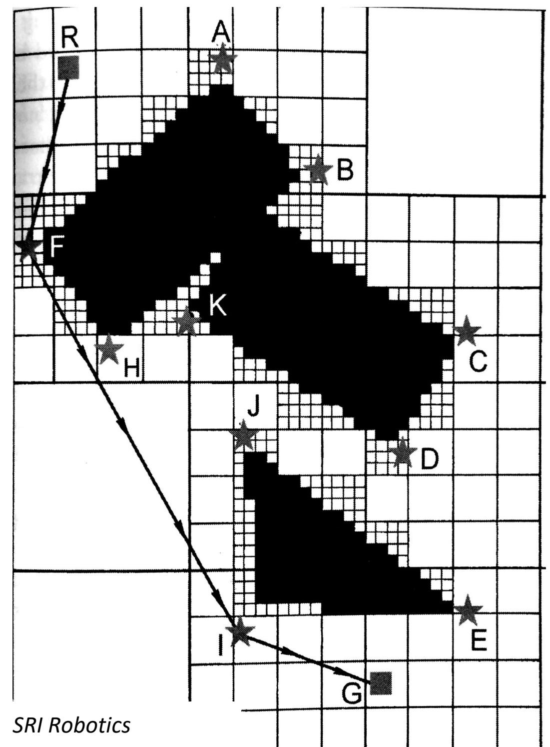
Estimate of the cost of the optimal path from node v to the goal: $h(v)$

A*

- The informed version of UCS is called A*
- Very popular search algorithm
- It was born in the early days of mobile robotics when, in 1968, Nilsson, Hart, and Raphael had to face a practical problem with Shakey (one of the ancestors of today's mobile robots)



Wikipedia



SRI Robotics

A*

- The idea behind A* is simple: perform a UCS, but instead of considering accumulated costs consider the following:

Heuristic ("cost-to-go") → $h(n)$

$$f(n) = g(n) + h(n)$$

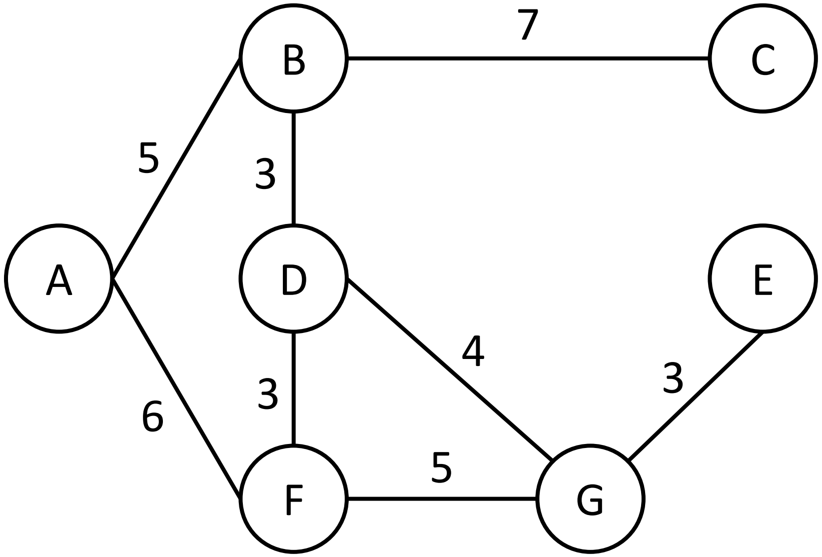
Cost accumulated on the path to n ("cost-to-come") → $g(n)$

- To guarantee that the search is sound and complete we need to require that the heuristic is **admissible**: it is an optimistic estimate or, more formally:

$$h(n) \leq \text{ Cost of the minimum path from n to the goal}$$

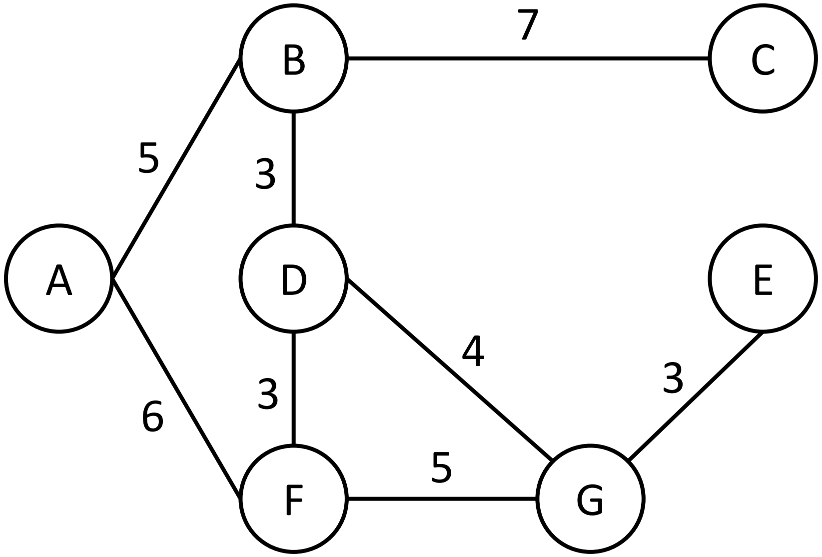
- If the heuristic is not admissible we might discard a path that could actually turn out to be better that the best candidate found so far

A*



node v	$h(v)$
A	10
B	7
C	1
D	3
E	0
F	7
G	2

A*



A
0+10=10

node v	$h(v)$
A	10
B	7
C	1
D	3
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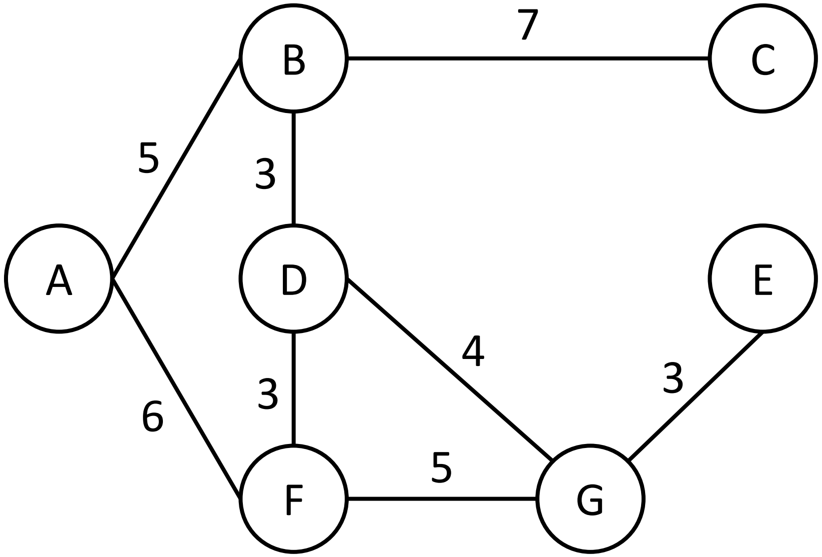
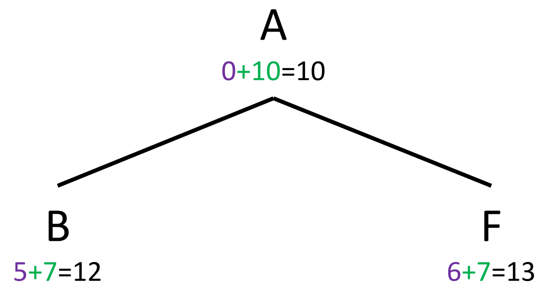
A*





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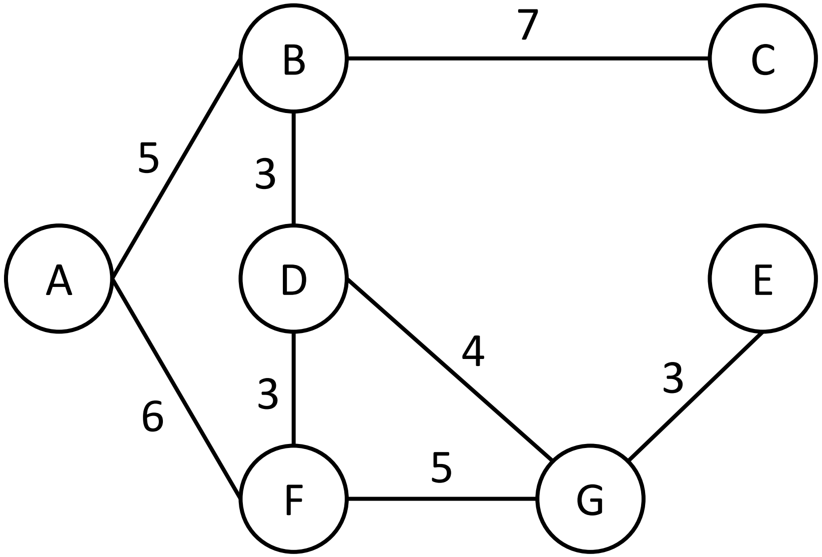
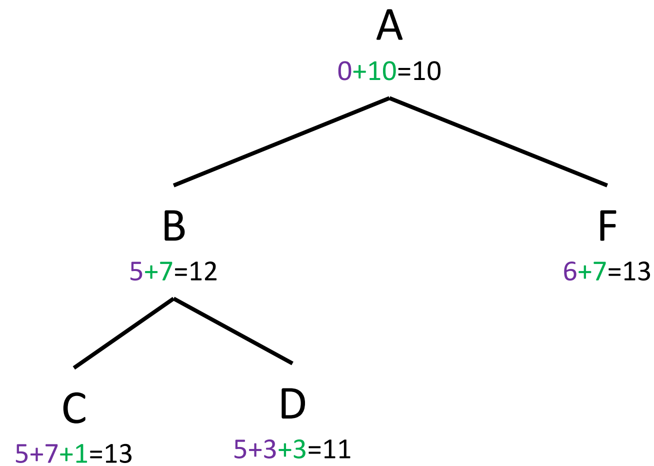
A*





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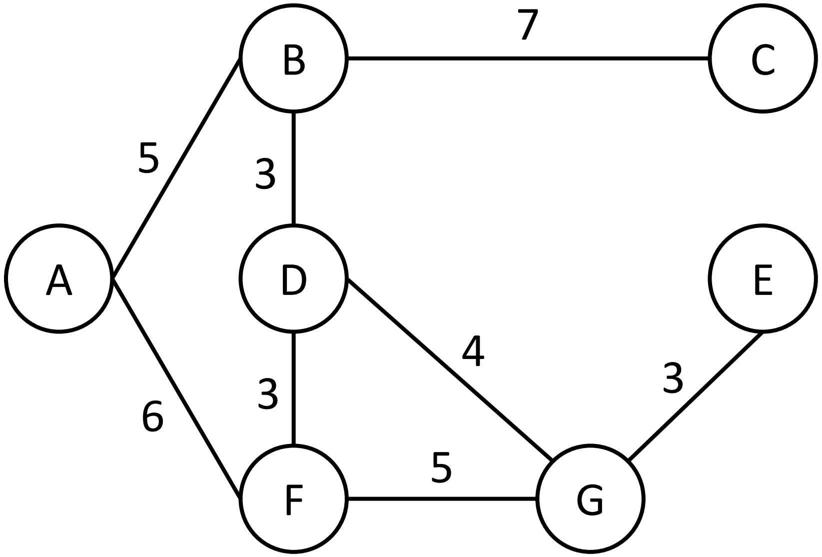
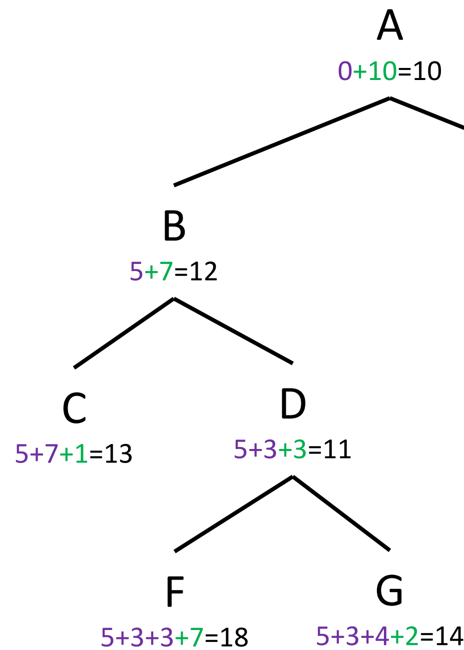
A*





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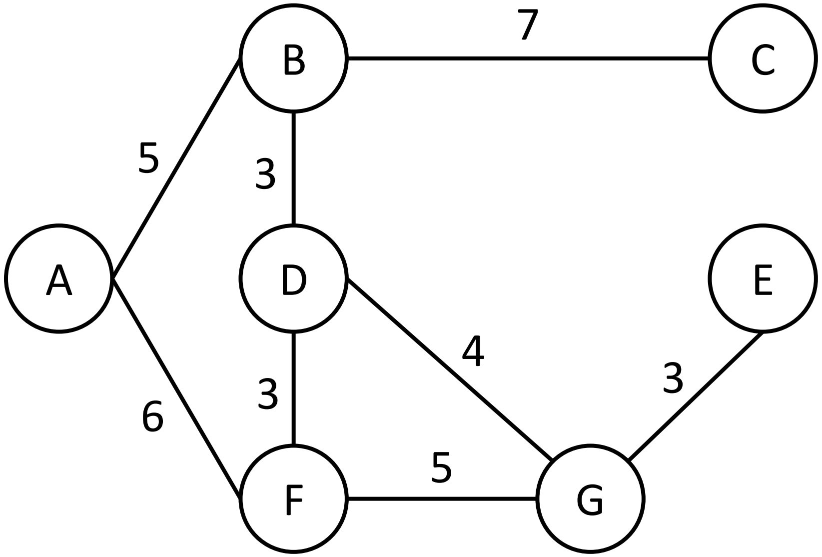
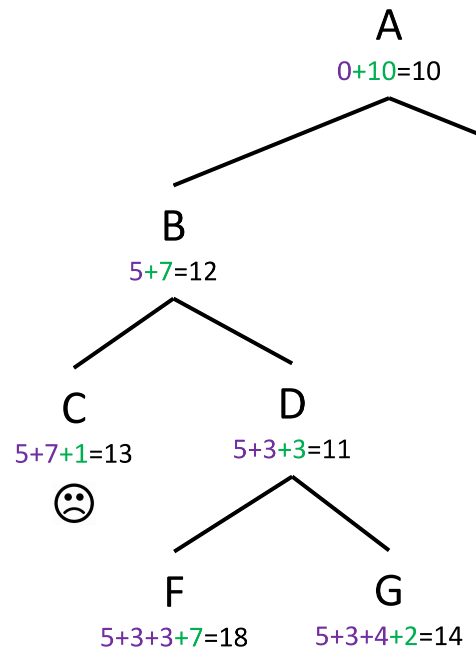
A*





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A	10
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C	1
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F	7
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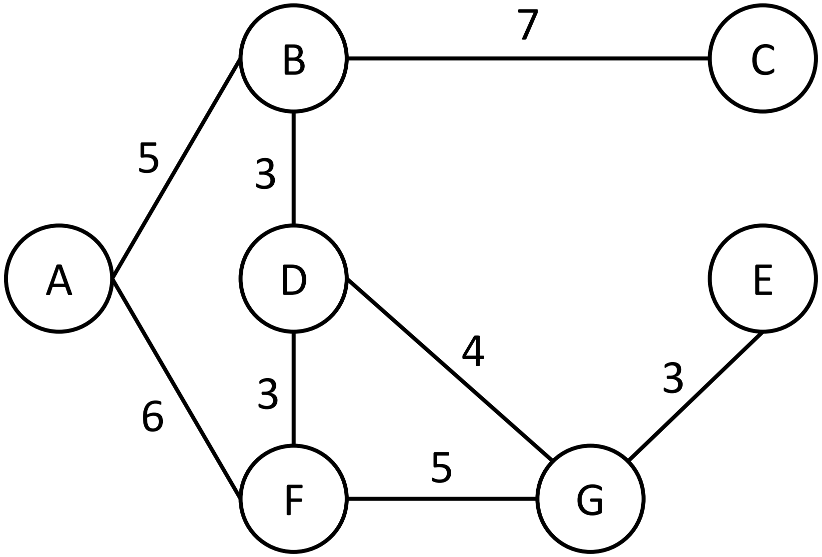
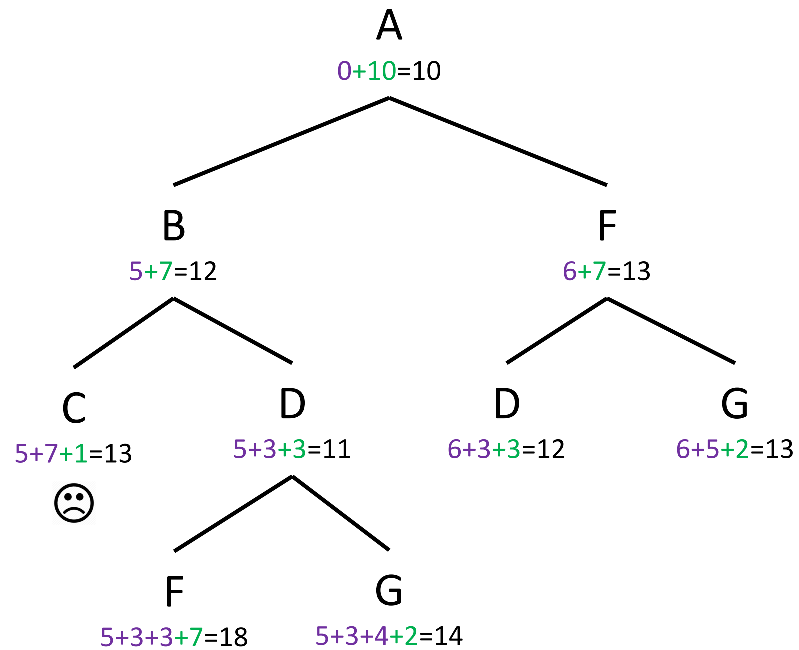
A*





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A	10
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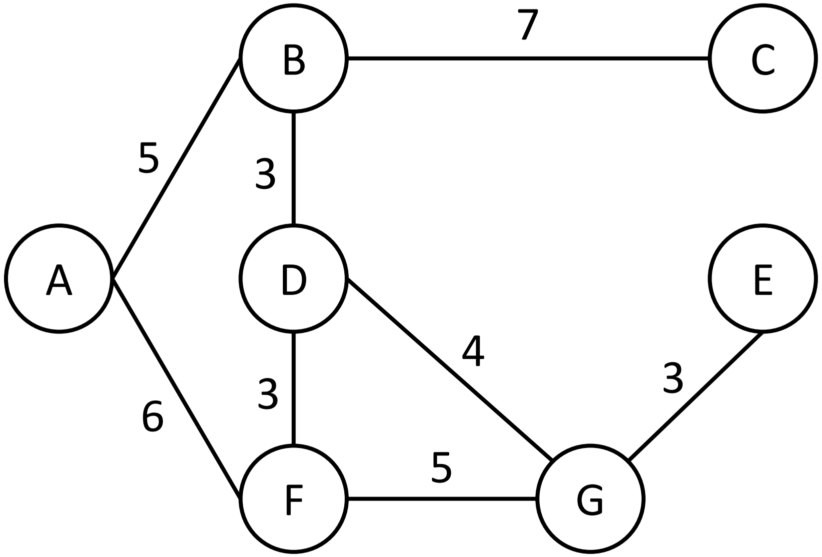
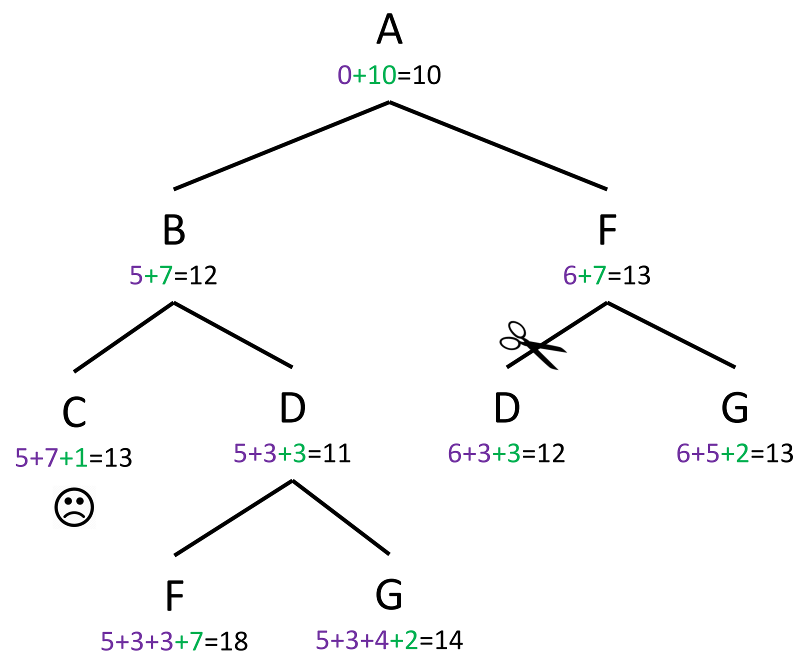
G	2

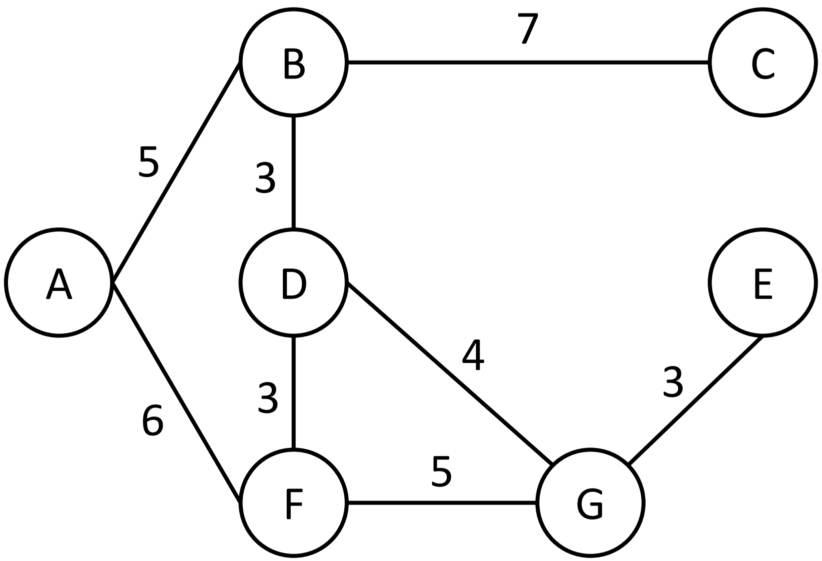
A*

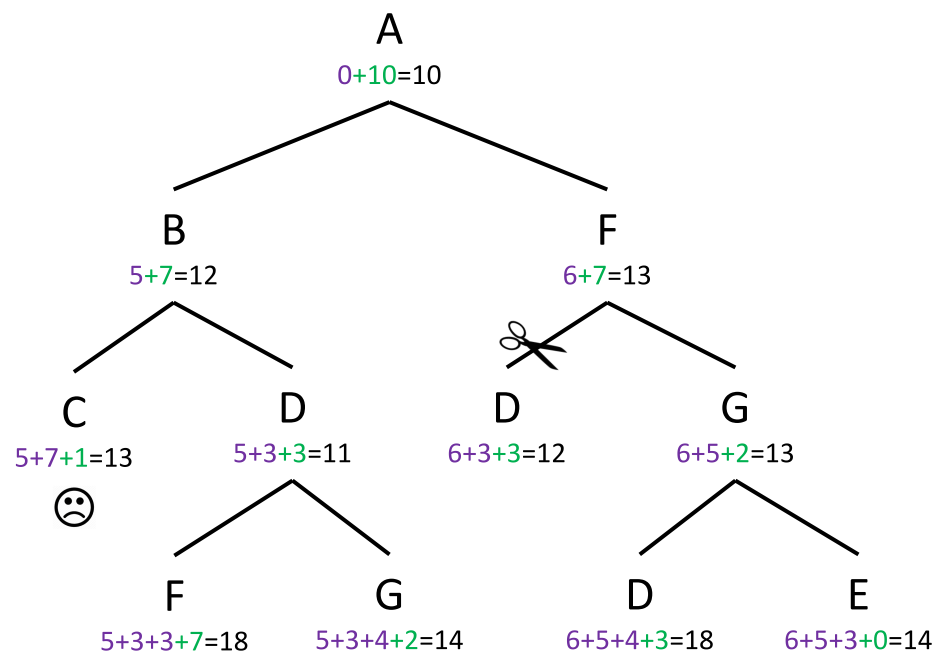




node v	$h(v)$
A	10
B	7
C	1
D	3
E	0
F	7
G	2

A*





node v	$h(v)$
A	10
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D	3
E	0
F	7
G	2

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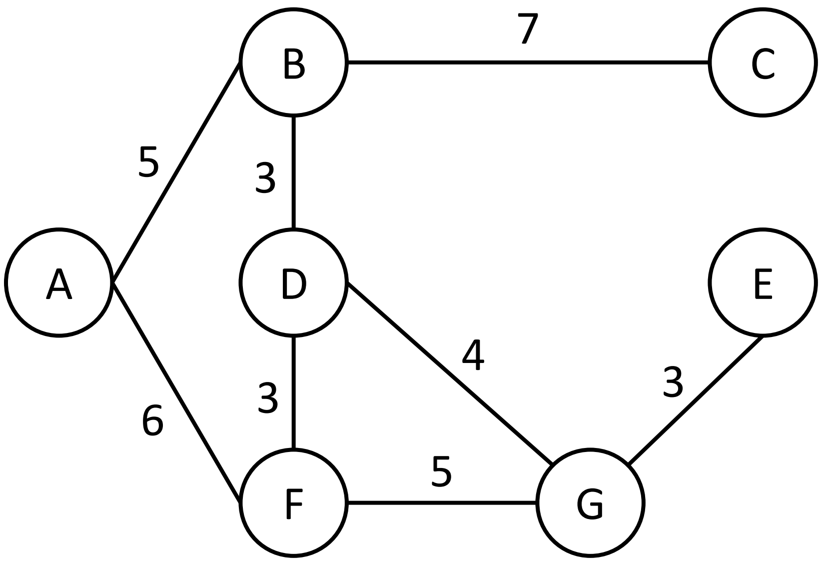

node v	$h(v)$
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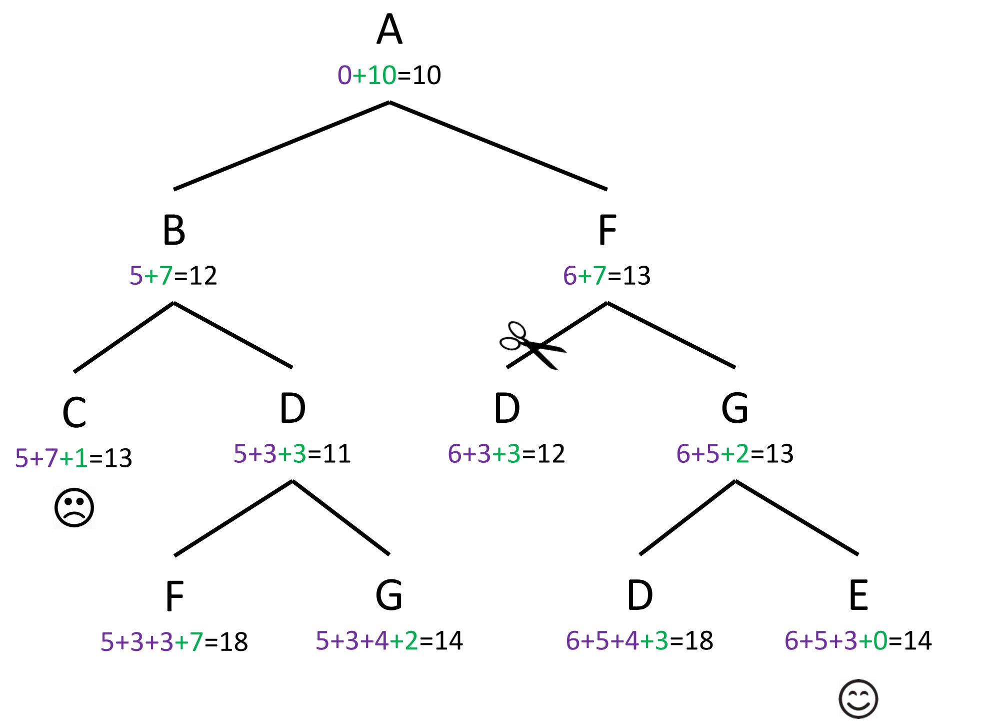
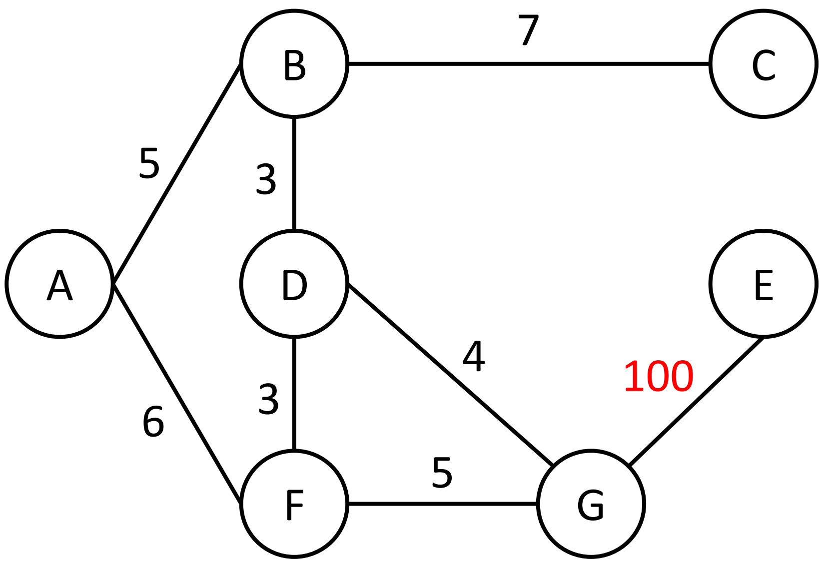
F	7
G	2



A*

- Problem: if we work with an extended list, admissibility is not enough!

- Let’s consider this “pathological" instance:



node v	$h(v)$
A	10
B	0
C	1
D	0
E	0
F	100
G	0

A*

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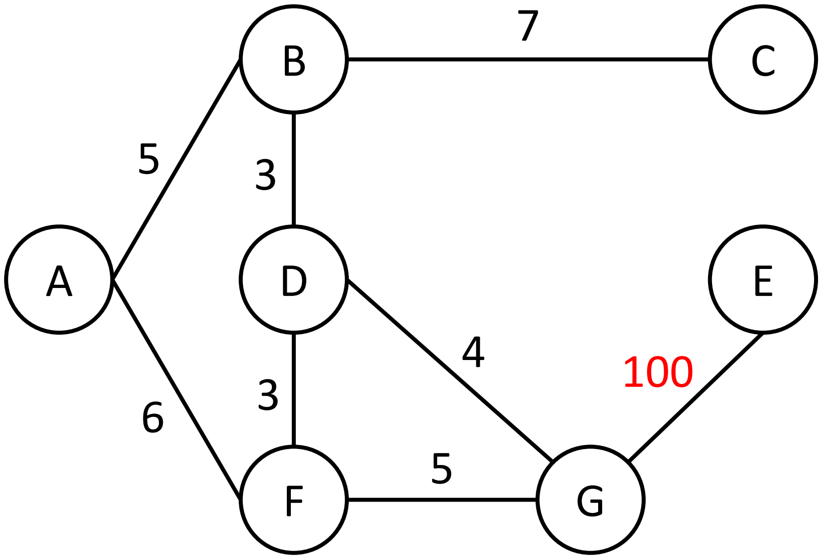

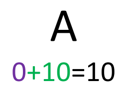

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E	0
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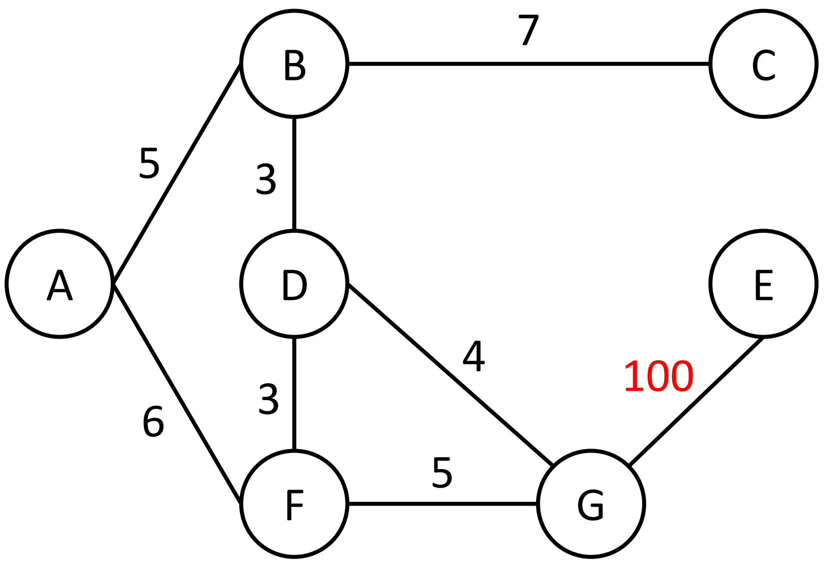

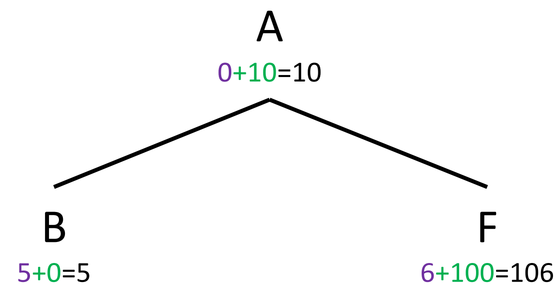

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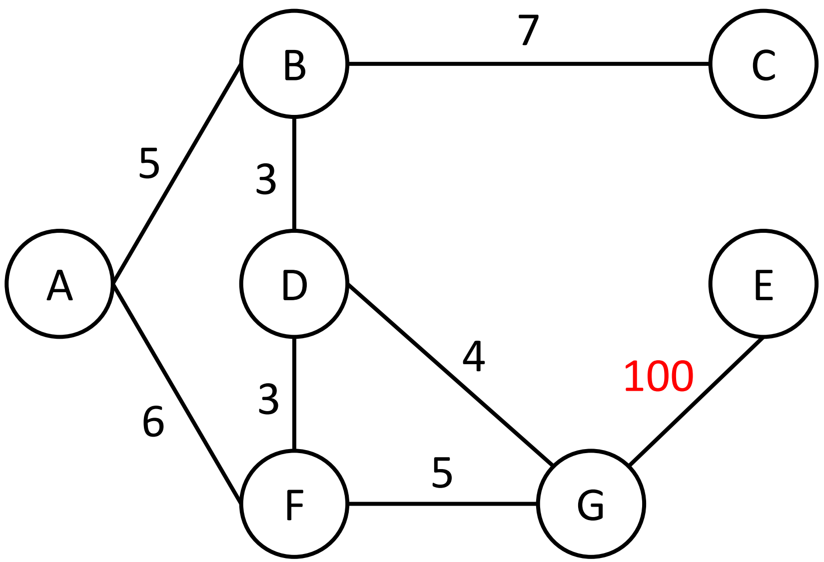

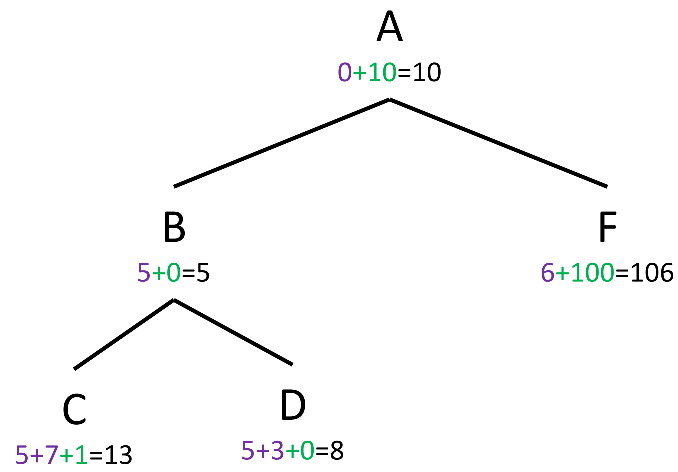

node v	$h(v)$
A	10
B	0
C	1
D	0
E	0
F	100
G	0

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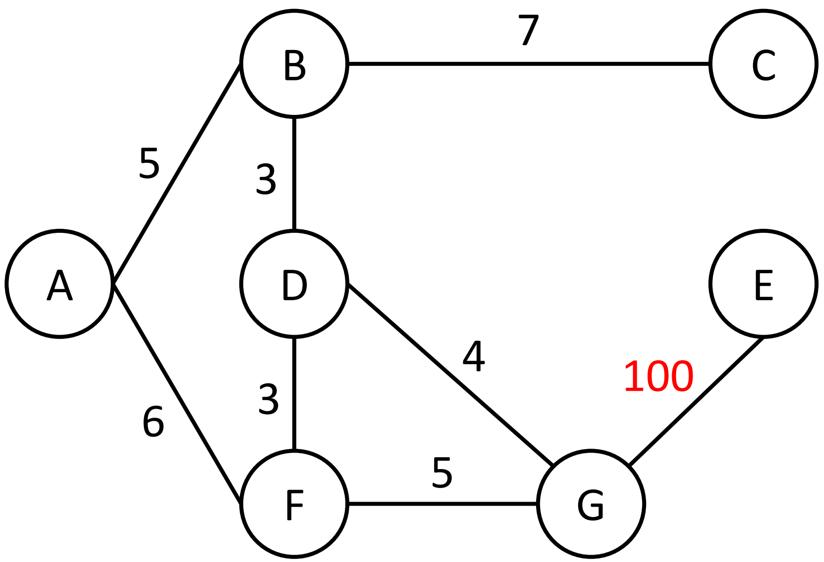

node v	$h(v)$
A	10
B	0
C	1
D	0
E	0
F	100
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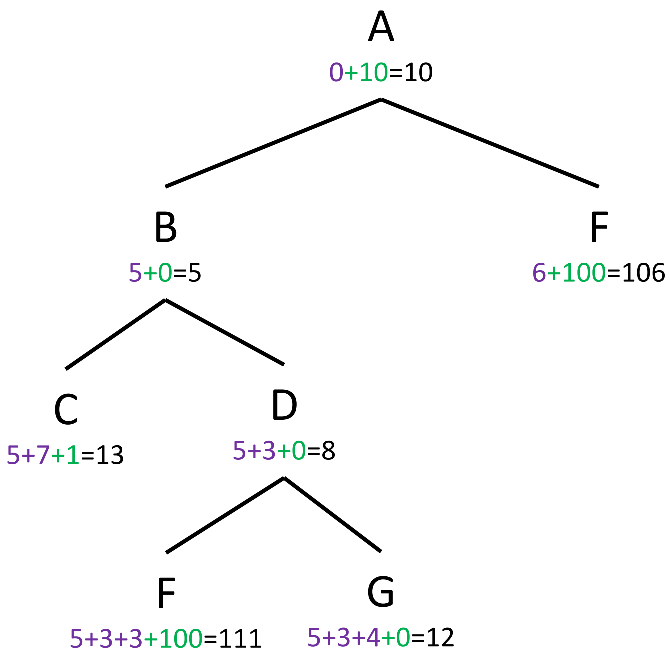

A*

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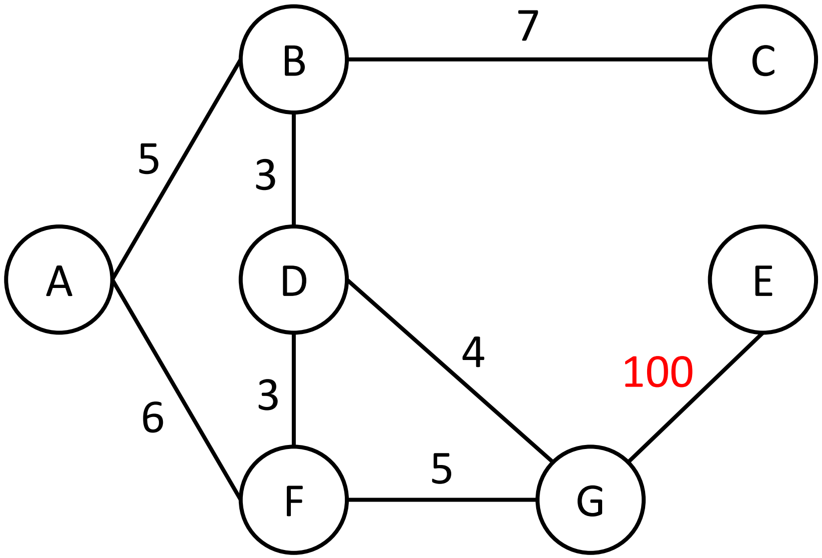
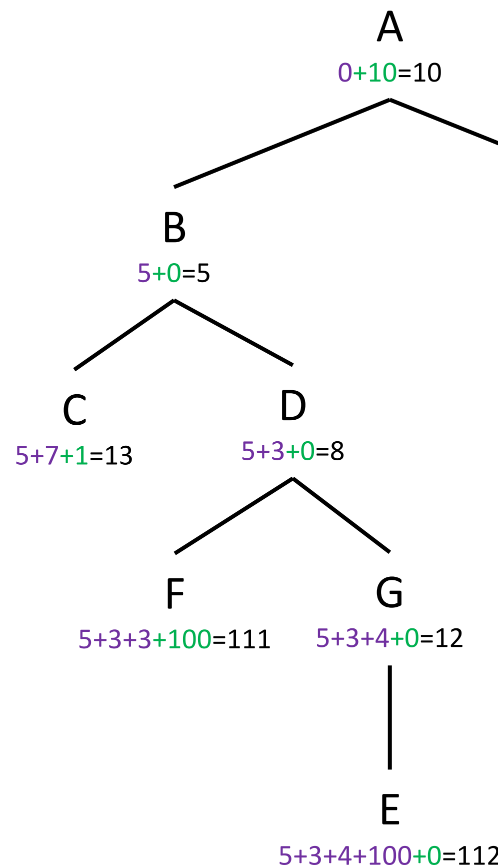


node v	$h(v)$
A	10
B	0
C	1
D	0
E	0
F	100
G	0



A*

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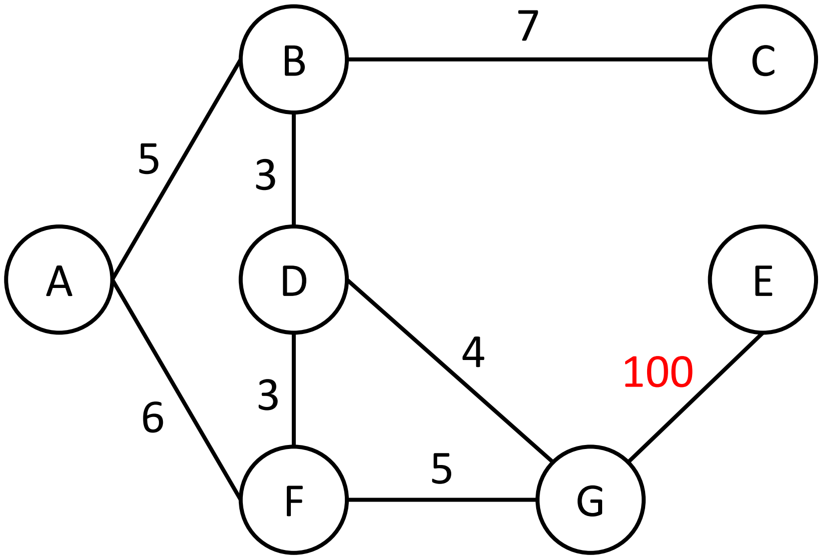

node v	$h(v)$
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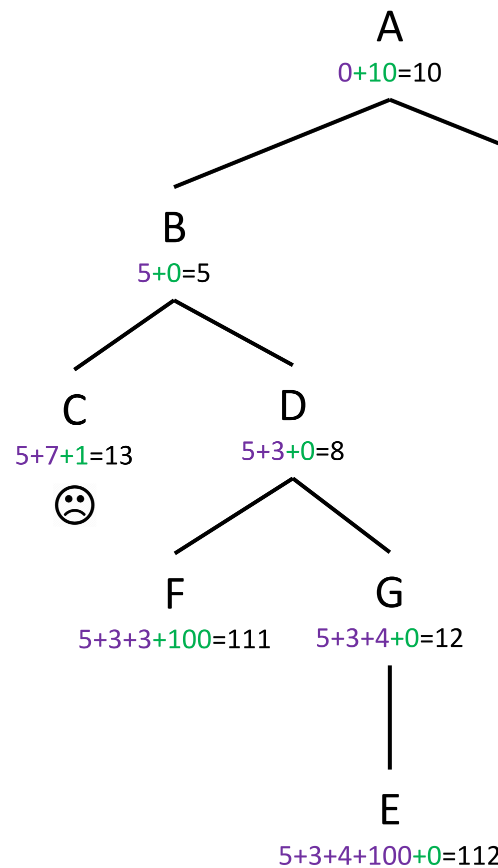
C	1
D	0
E	0
F	100
G	0



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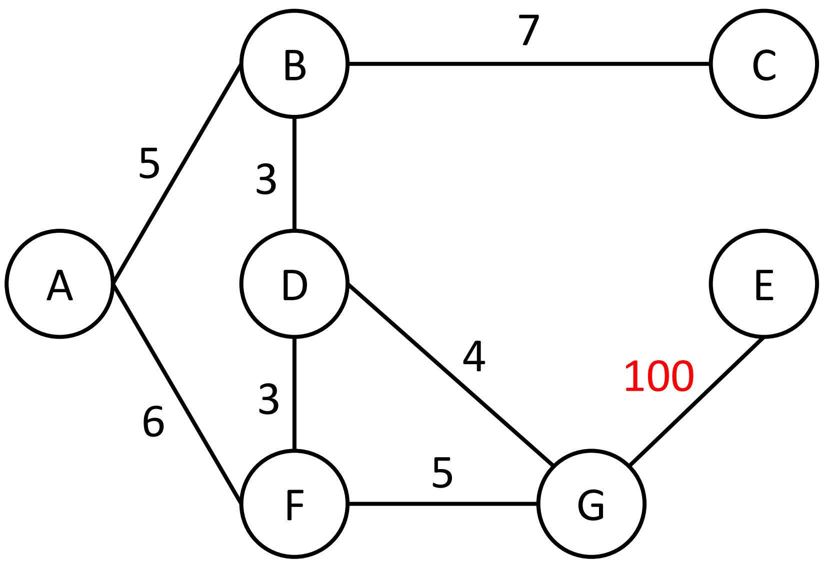
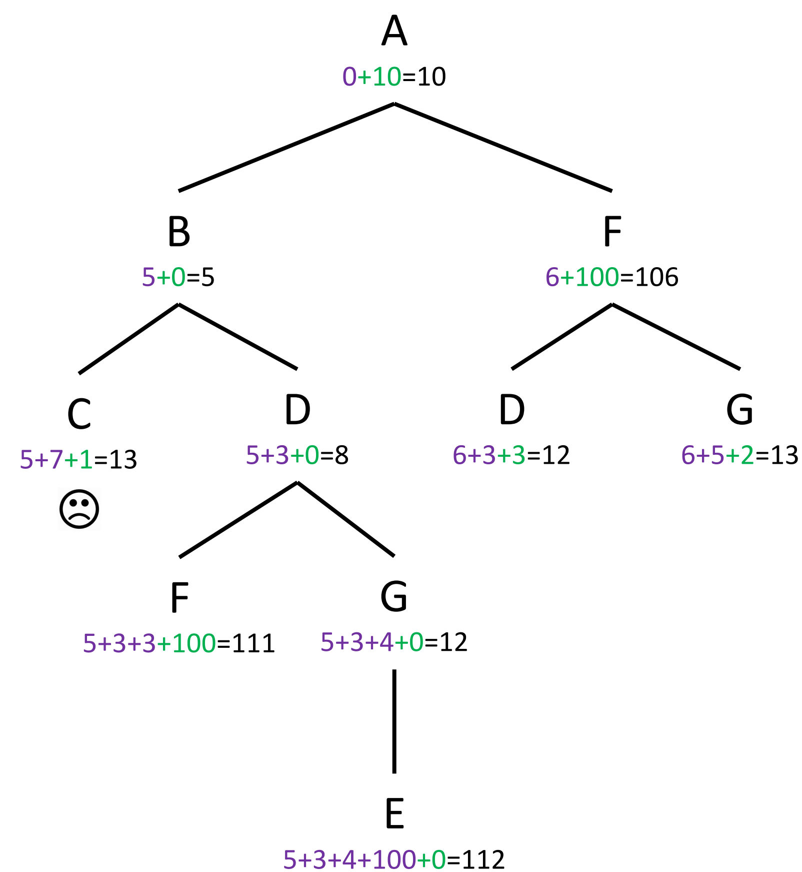


node v	$h(v)$
A	10
B	0
C	1
D	0
E	0
F	100
G	0



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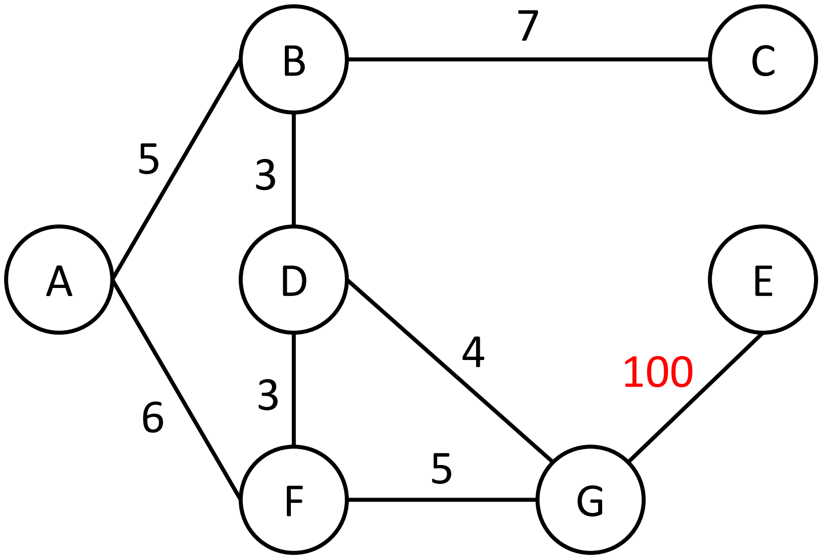
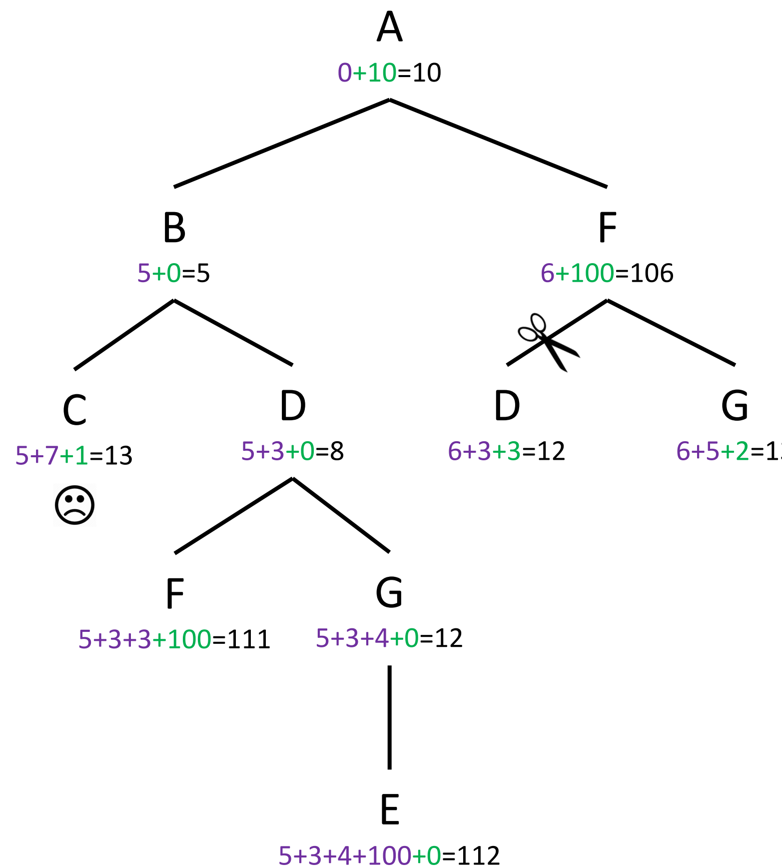


node v	$h(v)$
A	10
B	0
C	1
D	0
E	0
F	100
G	0



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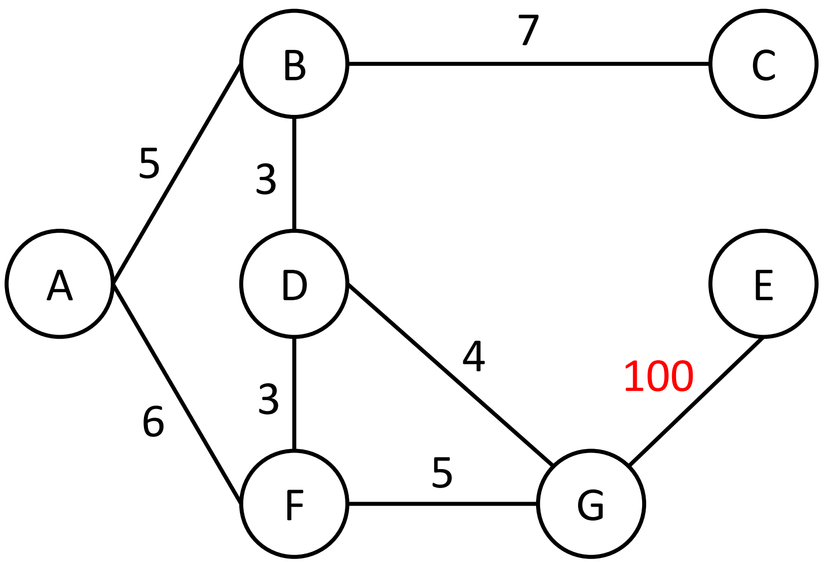

node v	$h(v)$
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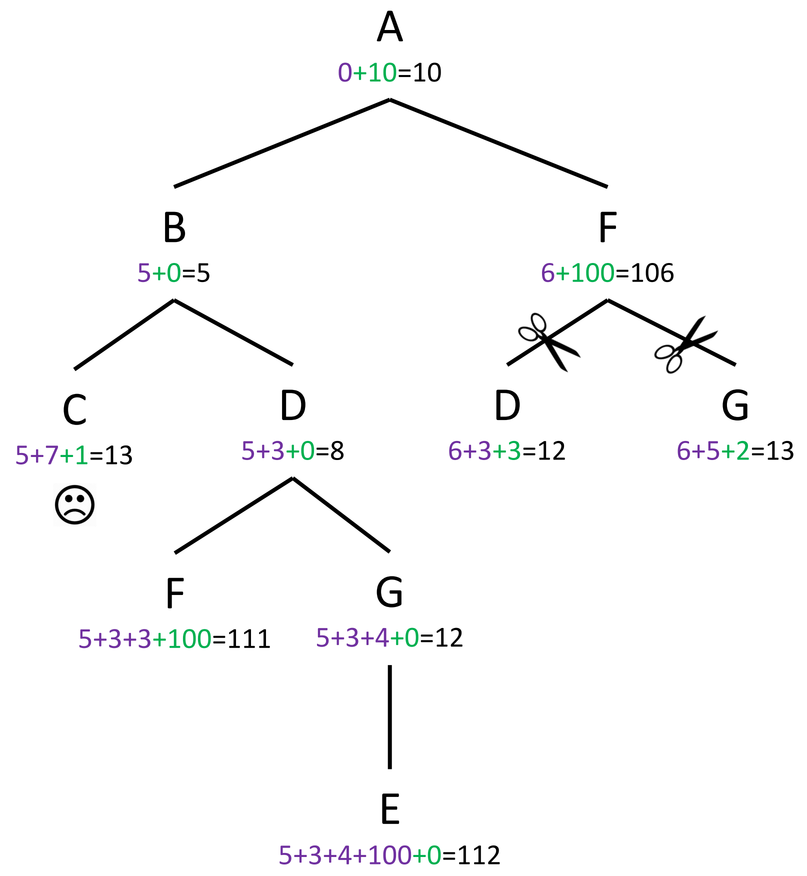
D	0
E	0
F	100
G	0



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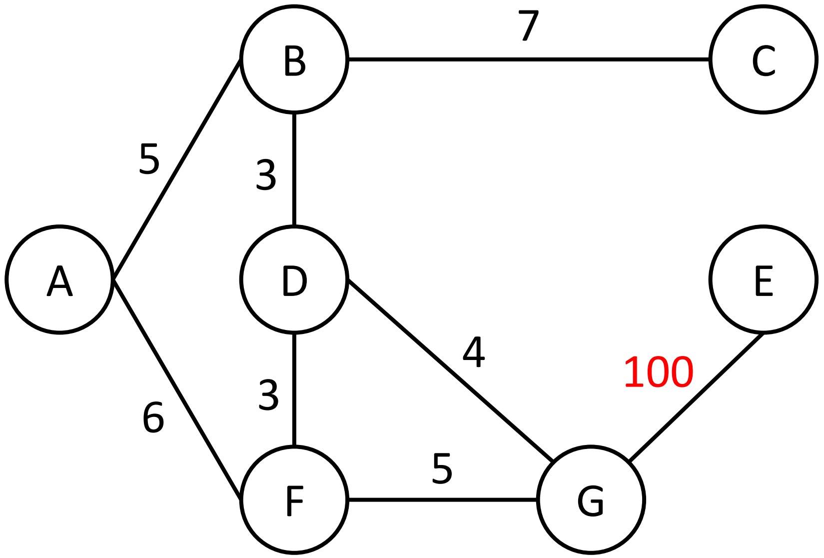

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F	100
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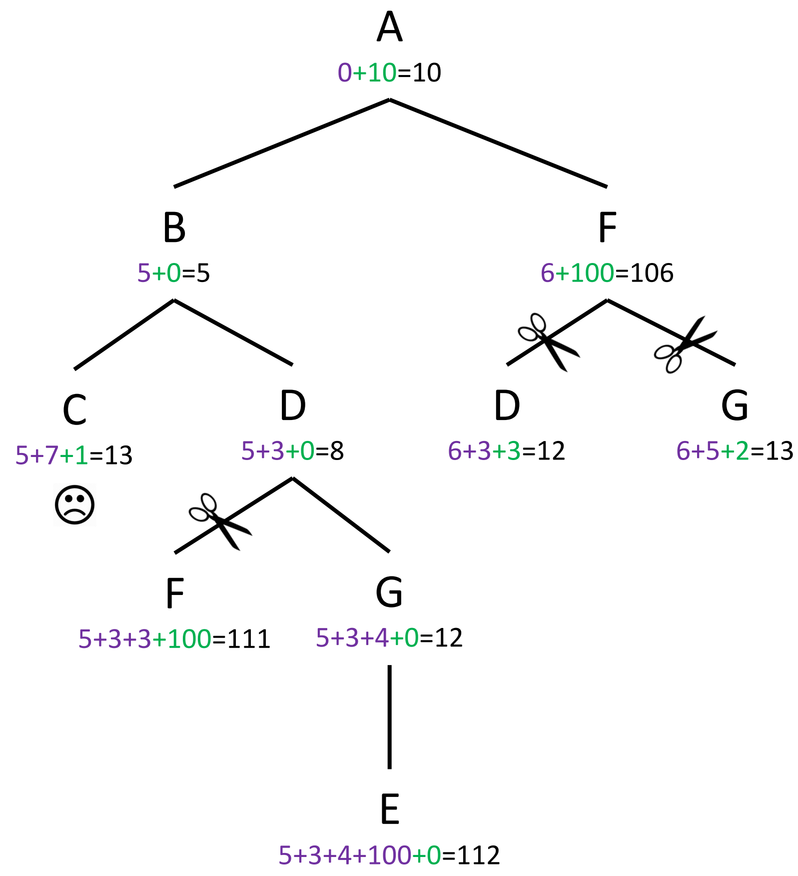

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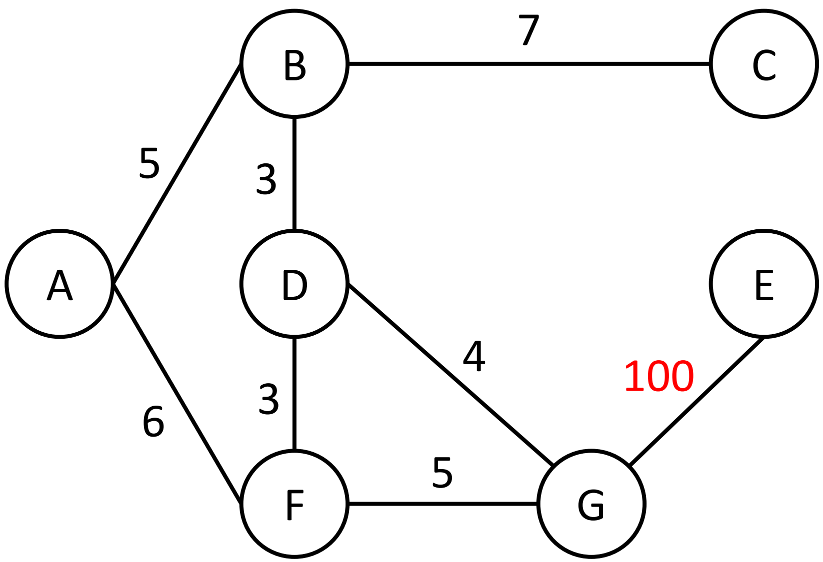

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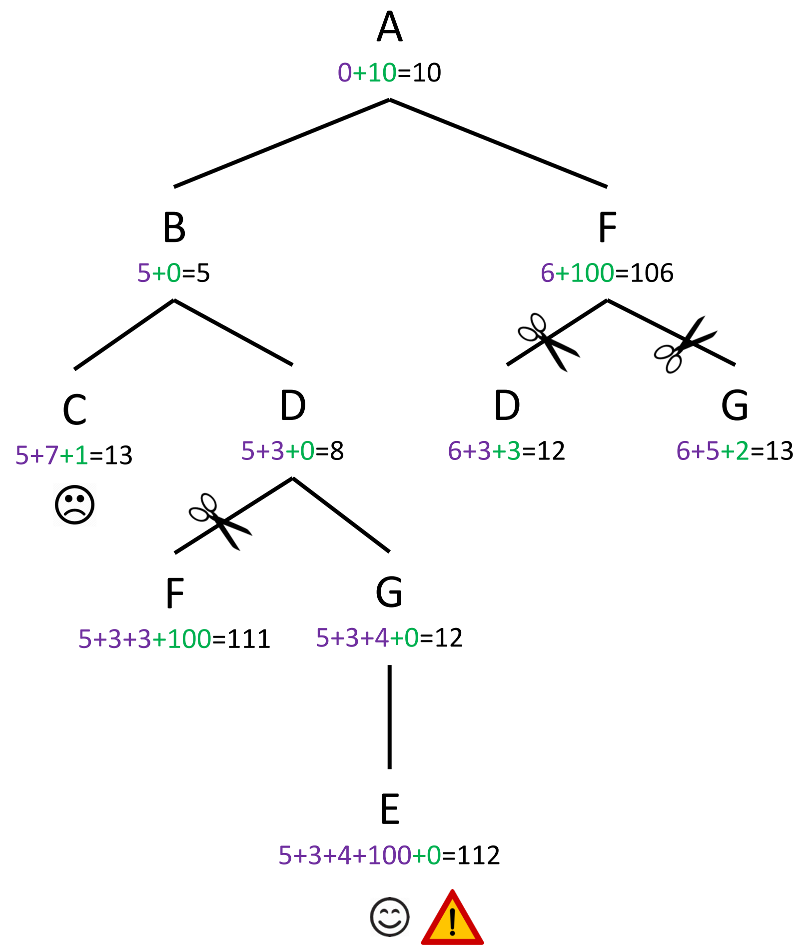
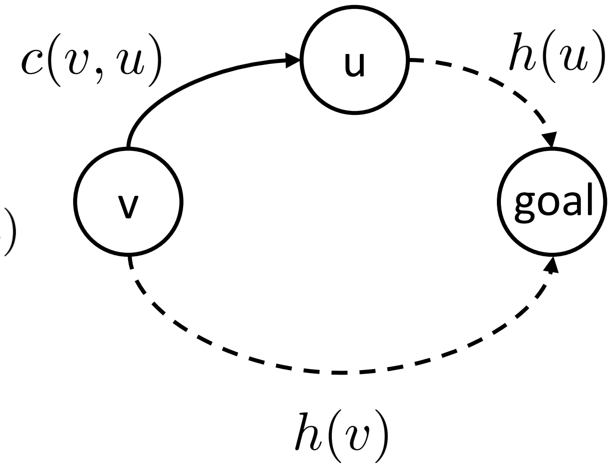
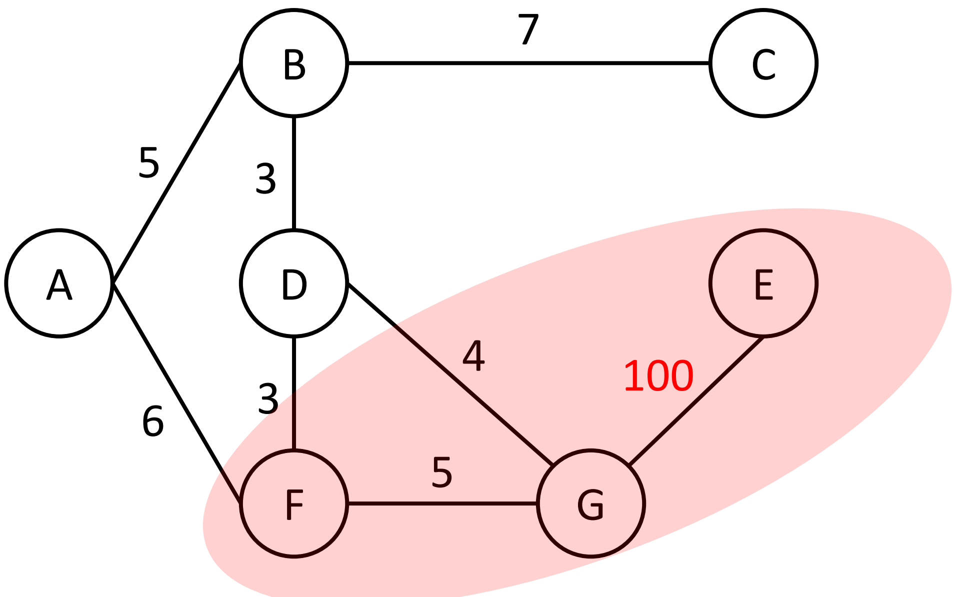


A*



- We need to require a stronger property: **consistency**
- For any connected nodes u and v: $h(v) \leq c(v,u) + h(u)$
- It's a sort of triangle inequality, let's reconsider our pathological instance:

node v	$h(v)$
A	10
B	0
C	1
D	0
E	0
F	100
G	0



Optimality of A*

$$f(v) = g(v) + h(v)$$

$$f(u) = g(u) + h(u) = g(v) + c(v,u) + h(u) \geq g(v) + h(v)$$

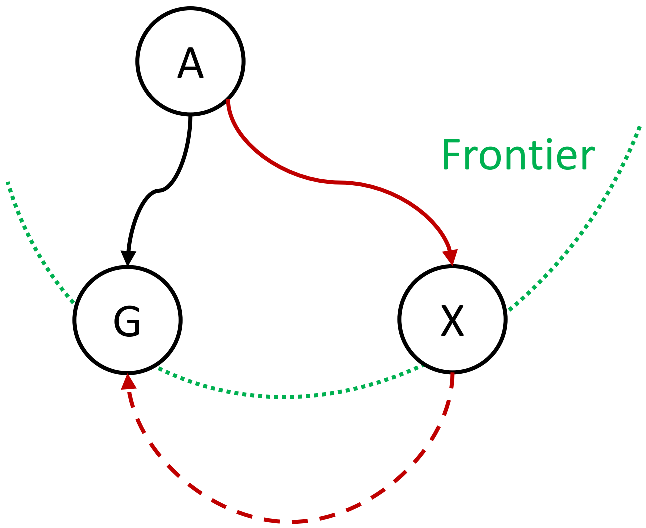
consistency

$f(u) \geq f(v)$ ⟶ f is non-decreasing along any search trajectory

Hypotheses:

1. A* selects from the frontier a node G that has been generated through a path p
2. p is not the optimal path to G

Given 2 and the frontier separation property, we know that there must exist a node X on the frontier that is on a better path to G



f is non-decreasing: $f(G) \geq f(X)$

A* selected G: $f(G) < f(X)$

When A selects a node for expansion, it discovers the optimal path to that node*

Building good heuristics

- A "larger" heuristic is better usually than a smaller one. The trivial heuristic is $h(v) = 0$.

- The "larger heuristics are better" principle is not a methodology to define a good heuristic

- Such a task, seems to be rather complex: heuristics deeply leverage the inner structure of a problem and have to satisfy a number of constraints (admissibility, consistency, efficiency) whose guarantee is not straightforward

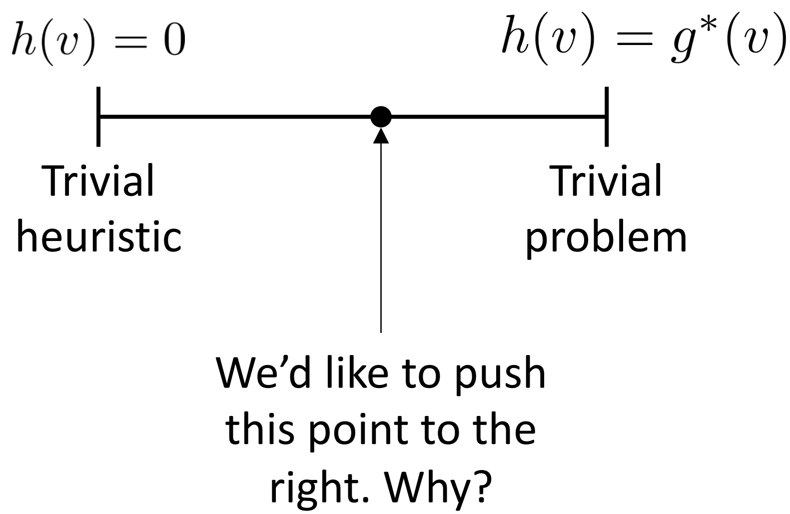
- When we adopted the straight-line distance in our route finding examples, we were sure it was a good heuristic

- Would it be possible to generalize what we did with the straight-line distance to define a method to *compute* heuristics for a problem?

- Good news: the answer is yes

Evaluating heuristics

- How to evaluate if an heuristic is good?



- A* will expand all nodes v such that: $f(v) < g^*(goal) \longrightarrow h(v) < g^*(goal) - g(v)$

- If, for any node v $h_1(v) \leq h_2(v)$

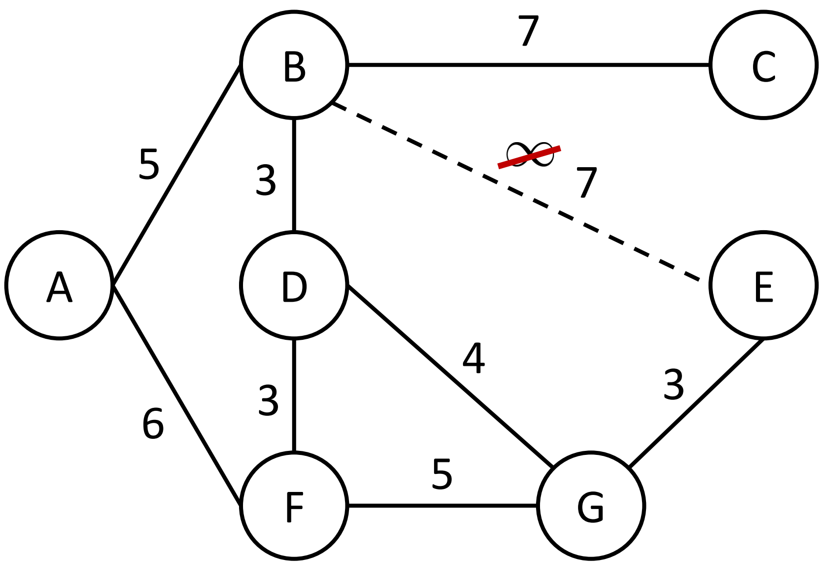
  then A* with h_2 will not expand more nodes than A* with h_1, in general h_2 is better (provided that is consistent and can be computed by an efficient algorithm)

- If we have two consistent heuristics h_1 and h_2 we can define

$$h_3(v) = \max\{h_2(v), h_1(v)\}$$

Relaxed problems

- Given a problem P, a relaxation of P is an easier version of P where some constraints have been dropped



$$\hat{g}(v,u) \leq g(v,u)$$

Costs in the relaxation

Costs in the original problem

- In our route finding problems removing the constraint that movements should be over roads (links) means that some costs pass from an infinite value to a finite one (the straight-line distance)

Relaxed problems

- Idea:

Define a relaxation of P: $\hat{P}$ → Apply A* to every node and get $\hat{h}^*(v)$ → Set $h(v) = \hat{h}^*(v)$ in the original problem and run A*

- We can easily define a problem relaxation, it's just matter of removing constraints/rewriting costs

- But what happens to soundness and completeness of A*?

$\hat{h}^*(v) \leq \hat{g}(v,u) + \hat{h}^*(u)$	Path costs are optimal
$h(v) \leq \hat{g}(v,u) + h(u)$	From our idea
$\hat{g}(v,u) \leq g(v,u)$	From the definition of relaxation
$h(v) \leq g(v,u) + h(u)$	**h is consistent**

Heuristics example: 8-puzzle

- How to evaluate if an heuristic is good?

7	2	4
5		6
8	3	1

Start State

	1	2
3	4	5
6	7	8

Goal State

- $h_1(v)$ the number of misplaced tiles
- $h_2(v)$ sum of distances of tiles from their goal destination (Manhattan Distance)
- $h_1(v) = 8$, $h_2(v) = 18$, $h_*(v) = 26$
- Both heuristics are admissible; the second one is “higher”, so is close to the actual cost of the optimal path. So it is a better heuristic.
- If we have two consistent heuristics h_1 and h_2 we can define

$$h_3(v) = \max\{h_2(v), h_1(v)\}$$

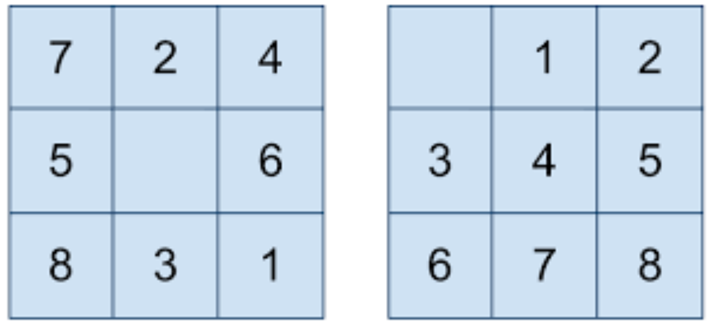
Heuristics example: 8-puzzle

	Search Cost (nodes generated)			Effective Branching Factor		
d	IDS	$A^*(h_1)$	$A^*(h_2)$	IDS	$A^*(h_1)$	$A^*(h_2)$
2	10	6	6	2.45	1.79	1.79
4	112	13	12	2.87	1.48	1.45
6	680	20	18	2.73	1.34	1.30
8	6384	39	25	2.80	1.33	1.24
10	47127	93	39	2.79	1.38	1.22
12	3644035	227	73	2.78	1.42	1.24
14	–	539	113	–	1.44	1.23
16	–	1301	211	–	1.45	1.25
18	–	3056	363	–	1.46	1.26
20	–	7276	676	–	1.47	1.27
22	–	18094	1219	–	1.48	1.28
24	–	39135	1641	–	1.48	1.26

- $h_1(v)$ the number of misplaced tiles
- $h_2(v)$ sum of distances of tiles from their goal destination
(Manhattan Distance)
- How to evaluate an heuristic? Compute several instances of the problem and compute the *effective branching factor*
(the number of branches expanded by the search strategy during search)
In the table we tested 1000+ instances of the problem.
- $h_2(v)$ dominates $h_1(v)$ and is 50k better wrt IDS with d=12

Heuristics example: 8-puzzle

- How to evaluate if an heuristic is good?



Start State   Goal State

Remember that the relaxed problem *adds edges* to the state space

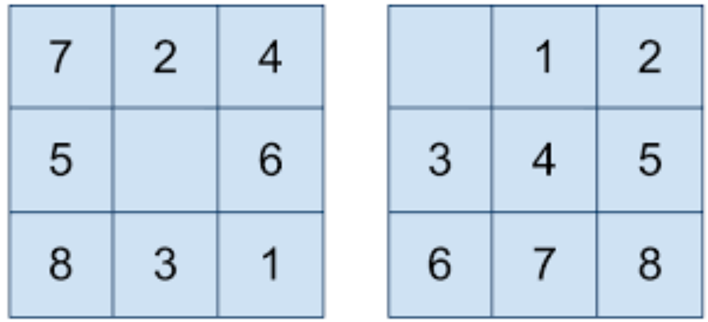
- any optimal solution in the original problem is, by definition, also a solution in the relaxed problem;
- however the relaxed problem may have *better* solutions if the added edges provide short cuts

Hence, *the cost of an optimal solution to a relaxed problem is an admissible heuristic for the original problem.*

Furthermore, because the derived heuristic is an exact cost for the relaxed problem, it must obey the triangle inequality and is therefore **consistent**

Heuristics example: 8-puzzle

- How to evaluate if an heuristic is good?



Start State Goal State

How to generate heuristics? We can remove rules / costraints

8:puzzle rules:
A tile can move from square A to square B if:
A is horizontally or vertically adjacent to B **and** B is blank.

we can generate three relaxed problems by removing one or both of the conditions:
(a) A tile can move from square A to square B if A is adjacent to B.
(b) A tile can move from square A to square B if B is blank.
(c) A tile can move from square A to square B.

References

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Sistemi Intelligenti Avanzati
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